Dear Editor-in-Chief Lily Maysari Angraini,

Thank you very much for your assistance in reviewing our manuscript. We take the comments from the reviewer seriously and respond to them carefully. Below we clarify the issues raised by the reviewers and modify our paper accordingly. We hope that the revised paper can be reconsidered in IPR. Thank you very much.

Best regards,

Adrianus Inu Natalisanto on behalf of all authors

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REVIEWER COMENT(S):

**Reviewer: 1**

COMMENTS TO THE AUTHOR(S)

► Object: **Analysis and Discussion**

 Obtaining eq. 37 is really doubtful. It means the whole result until the very end needs to be checked. Please give me the explanation why you get eq. 37?

**Our response:** We thank for the reviewer’s comments. Equation 37 (Eq. 39 in the revised manuscript) arises in the context of light propagation through a medium with a spatially varying refractive index. In such cases, the trajectory of light deviates from a straight line due to the gradient of the refractive index, leading to continuous refraction along the path. Mathematically, this bending of the light path (see Figure 2 and 3 in the manuscript) can be described by the Frenet-Serret equations from vector analysis, which govern the evolution of the tangent, normal, and binormal vectors along a space curve. Equation 37 encapsulates this geometric behavior in the context of light trajectories under non-uniform optical conditions.

► Object: **Special Notes**

1. Please use [1-6] instead of [1][2][3][4][5][6]. For [1][2][3][5][6][7][8] please use [1-3,5-8]
2. In the manuscript, the word ‘author’ is repeated. One should use: … in ref. [20], it used …
3. eq. 7, 8, 10 (other equations too) need to be written straightforwardly.
4. English needs to be improved
5. The claim of the author of the paper is rather exaggerated, e.g. This study introduces a groundbreaking reformulation of geometrical optics…. Please consider changing it.
6. The acknowledgment needs to be removed. Nothing is to be written here according to the ‘written’ made by the author.

**Our response:** We thank for the reviewer’s suggestions.

1. We have revised the citation style throughout the manuscript to use the requested format, such as [1–6] and [1–3, 5–8], in accordance with the reviewer’s comment.
2. We have revised the manuscript to avoid the repetitive use of the word “author.” In the revised version, we now use more objective and concise phrasing, such as “… in ref. [20], it was used …,” as recommended.
3. We have revised equations 7, 8, 10, and the others to be written more clearly and straightforwardly, as requested by the reviewer.
4. We have revised the manuscript to improve the English language as requested by the reviewer.
5. We appreciate the reviewer’s comment regarding the tone of the manuscript. We agree that the original phrasing may have overstated the contribution. Accordingly, we have revised the sentence to read: “This study revisits the formulation of geometrical optics and proposes a possible refinement.” We believe this more accurately reflects the scope and intention of the work.
6. The acknowledgment section has already been removed. Nothing is written there as per the author's guidelines.

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**Reviewer: 2**

COMMENTS TO THE AUTHOR(S)

► Object: **Introduction**

1. What is the significance of this reformulation?
2. How does the numerical simulation provide insightful results regarding the reformulation?
3. In the text, the refractive index depends on **r.** However, it is mentioned that n(**r**) is approximated by some value, which does not make sense unless it becomes a constant. Please provide a clarification.

**Our response:** We thank for the reviewer’s comments.

1. The significance of this reformulation lies in its effort to reinterpret geometric optics within the framework of Abelian U(1) gauge theory, allowing the derivation of phase equations and geodesic light trajectories in various media. By associating the optical phase with gauge potentials and interpreting refractive index variations as analogs of gauge fields and field strength tensors, this approach provides a mathematically consistent and physically accurate framework. The corrected formulation ensures compliance with fundamental physical behaviors, such as linear light propagation in vacuum and correct bending in graded-index and metamaterial environments. Supported by numerical simulations, the framework offers a promising avenue for modeling optical phenomena, especially phase shifts and light bending, in homogeneous, anisotropic, and negative-index materials.
2. The numerical simulations provide critical insights and serve as a validation mechanism for the reformulated theory. Simulations in vacuum and homogeneous media reproduce the expected straight-line trajectories of light rays, confirming that the theory aligns with established physical behavior. More importantly, simulations in anisotropic materials and optical metamaterials demonstrate how the gauge potentials and field strengths affect ray trajectories and phase evolution. These results not only confirm the applicability of the reformulation to complex media but also offer predictive capabilities for future experimental setups. Additionally, the simulation outcomes reveal the potential for describing optical phase shifts and topological effects, which are essential for applications in interferometry, optical metrology, and fiber-optic communications.
3. To address the reviewer's question regarding the refractive index n(**r**), we can clarify that the refractive index can indeed be spatially varying, meaning it depends on the position **r** within a medium. In the context of the article, the refractive index is treated as a function n(**r**) to account for gradients in the material's optical properties, such as in anisotropic media or optical metamaterials, where the refractive index can change with spatial location.

However, if the refractive index is approximated by a constant value in certain cases (as mentioned in the article), it is an idealization used to simplify the problem, particularly when considering homogeneous media or when the variations in n(**r**) are negligible over the scale of interest. In such cases, n(**r**) can be treated as approximately constant, which is a common approach in geometric optics to model simpler systems (e.g., light traveling through air or vacuum).

For more complex scenarios, such as in media with significant gradients in the refractive index (e.g., in optical metamaterials, anisotropic media, or media with varying properties), n(**r**) must be considered as a function of position to accurately describe the light's behavior, including refraction and phase shifts.

To summarize, the refractive index n(**r**) may be approximated by a constant value in simplified cases, but for more accurate modeling of light propagation in non-homogeneous media, it is essential to treat it as a spatially dependent function. The approximation to a constant should be explicitly stated to avoid confusion in cases where the refractive index does not significantly vary in space.

► Object: **Methods**

1. What is the justification for the refractive index model used in the paper?
2. Vector notation is not consistent in the manuscript.
3. Integration in Eq. (20) is not clear as we don’t know the relationship between l and **r**.
4. Eqs. (26) and (28) do not seem compatible.
5. What are the assumptions that justify Eq. (29) holds? How does it fit in your case? Please provide a justification.

**Our response:**

1. We thank for the reviewer’s comments. The refractive index model in this paper is developed within the framework of Abelian U(1) gauge theory, which is reformulated to describe geometrical optics under weak-field conditions. The justification for this model is based on the following considerations:

**First, Consistency with Classical Geometrical Optics**

The initial formulation of the phase equation presented in Ref. [11] contained a mathematically unclear representation of the refractive index and its relation to phase. In our study, this has been corrected to align with the well-established definition of optical path length in classical optics. Specifically, Equation (3) has been refined to Equation (21), ensuring physical clarity and compatibility with standard optics references [1–3].

**Second, Gauge-Theoretic Interpretation of the Refractive Index**

Within the gauge-theoretic framework, the refractive index *n(r)* is treated analogously to a gauge potential, while its spatial gradient **∇**n(**r**) acts as the field strength tensor. This interpretation allows light rays to be treated as analogous to charged particles moving under an optical force, as reflected in the derived geodesic equation (Eq. 42 in the revised manuscript).

**Third, Validation through Numerical Simulations**

The model has been validated via numerical simulations involving different types of optical media: homogeneous media, vacuum, anisotropic materials, and optical metamaterials. The resulting light trajectories and phase shifts conform with established physical behavior: straight-line propagation in homogeneous and vacuum media, directional changes in anisotropic media, and exotic behaviors such as negative refraction in metamaterials (Figures 4 and 5).

In summary, the refractive index model used in our study is not only physically justified and mathematically consistent but also provides a novel and versatile framework for understanding light propagation in both conventional and complex media within a gauge-theoretic context.

1. Thank you for pointing this out. We have carefully reviewed the entire manuscript to ensure that all vector quantities are written using a consistent notation. Specifically, we now consistently use **boldface letters** (e.g., **r**, **k**, **E**, **B**) to represent vector quantities throughout the text, equations, and figure captions. Scalar quantities are kept in regular italic. We appreciate the reviewer’s attention to detail, which has improved the overall clarity of our presentation.
2. Thank you for your valuable comment. We thank the reviewer for pointing out the lack of clarity in the integration process in Eq. (20). In our formulation, the parameter *l* represents the optical path length, which corresponds to the arc length along the light ray trajectory from vector position **r1** to **r**2​. Mathematically, this arc length satisfies the relation:

This means that *dl*=∣d**r**∣, where **r** is the position vector along the light path. Consequently, the integration in Eq. (20):

can be interpreted as a line integral over the trajectory of light, with *dl*=∣d**r**∣, consistent with the optical path length definition. We have included this clarification in Figure 2 to improve the physical and mathematical interpretation of the equation.

1. We thank the reviewer for the insightful observation regarding the apparent inconsistency between Eqs. (26) and (28). We acknowledge that, at first glance, these equations may appear incompatible. However, we would like to clarify that Eq. (26) is actually derived directly from Eq. (28), and thus they are mathematically consistent. For clarity and transparency, we provide the full derivation below:
* Starting from Eq. (28): **τ**=d**r**/d*l* ​ which implies τ d*l*=d**r**;
* Multiplying both sides by **τ**: **τ⋅τ** d*l*=**τ⋅**d**r**;
* Since **τ⋅τ**=1, this gives: d*l*=**τ⋅**d**r**;
* Taking the variation on both sides: δ(d*l*)=δ(**τ⋅**d**r**);
* Using the product rule for variations: δ(d*l*)=**τ⋅**δ(d**r**)+δ(**τ**)⋅d**r**;
* Noting that δ(**τ**)=0 at the extremal path (as per variational principle), we obtain: δ(d*l*)=**τ⋅**δ(d**r**) which is Eq. (26).

This derivation confirms that Eq. (26) is a logical consequence of Eq. (28), and the two equations are indeed compatible.

1. Thank you for your insightful and constructive question. Below, we provide an explanation of the assumptions underlying the validity of Equation (29) and its relevance in the context of our study.

Equation (29), , is derived under the following key **assumptions**:

* 1. **Applicability of Fermat’s Principle (Principle of Least Optical Path)**:
	The derivation assumes that light propagates along a path that makes the optical phase stationary. This is a direct analogue to the principle of least action in mechanics and is justified in geometrical optics for media with smoothly varying refractive indices.
	2. **Smooth and Differentiable Refractive Index Field *n(r)*:**

The function *n(r)* is assumed to be continuously differentiable, allowing for the computation of both its gradient **∇***n(r)* and its total derivative along the path d*n(r)*/d*l*​.

* 1. **Path Parametrization by Arc Length *l*:**

The parameter *l* is chosen as the arc length of the light path, implying |τ|=1, where **τ** is the unit tangent vector of the light ray trajectory.

* 1. **Variation of the Path within the Medium:**

The variation δ**r** is taken such that the endpoints *r1​* and *r2​* are fixed. This is standard in variational calculus when deriving geodesic-type equations.

**Fit in Our Case:**

Equation (29) plays a central role in our formulation of light propagation within the gauge-theoretic framework. It emerges from applying the calculus of variations to the corrected phase function , as detailed in Equations (22)–(28) of the manuscript. In our case, the assumptions mentioned above are satisfied because:

1. The refractive index functions used in simulations (both constant and spatially varying) are smooth and physically realizable.
2. The light paths are modeled as curves in 3D space parameterized by arc length *l*.
3. The variation is conducted in line with Fermat’s principle, aiming for stationary optical phase accumulation.

Equation (29) thus serves as a variational condition whose satisfaction leads to the geodesic equation of light rays in a medium with varying refractive index (see Eq. 42 in the revised manuscript), ensuring physical consistency and alignment with established optical principles.

► Object: **Analysis and Discussion**

1. What numerical method is used in Eqs (41), (42), (43)? Please provide a justification for all equations.
2. How are these numerical results justified by other results? Please provide references that can be used to validate the relevance of your results.

**Our response:**

1. We thank the reviewer for their insightful question regarding the numerical methods employed in Equations (41) (Eq. 45 in the revised manuscript), (42) (Eq. 48 in the revised manuscript), and (43) (Eq. 43 in the revised manuscript). Below, we clarify the derivation and numerical approaches for each equation, including appropriate justifications:

**Equation (41)** (Eq. 45 in the revised manuscript)

Equation (41) is derived from Equation (39) (Eq. 41 in the revised manuscript) (the ray deflection equation) by simplifying Equation (39) into (the derivative of the refractive index with respect to position) and implementing the central difference approximation (a common finite difference method used to numerically estimate derivatives). This approach allows for the discretization of the refractive index gradient, which is essential for simulating ray bending in inhomogeneous and anisotropic media.

**Equation (42)** (Eq. 48 in the revised manuscript)

Equation (42) is derived from Equation (40) (Eq. 42 in the revised manuscript), the optical geodesic equation, by applying

 , , and .

These equations represent the numerical discretization of spatial derivatives and curvature terms, also utilizing the **central difference method** for approximating the derivatives of the refractive index field. This method provides a practical and accurate means for computing light ray trajectories in media with complex refractive index distributions.

**Equation (43)** (Eq. 43 in the revised manuscript)

Equation (43) is obtained from Equation (17) by applying . This leads to a computational expression for the optical phase shift based on the optical path length. The numerical method used for solving this equation involves direct substitution of known values of the refractive index profile along the path of propagation.

In summary, the **central difference method**, a second-order accurate finite difference scheme, is the primary numerical technique used in the derivation and simulation of Equations (41) and (42), while Equation (43) is obtained through numerical evaluation of an integral using discretized refractive index values along the propagation path. All numerical implementations follow established procedures as outlined in Sastry (Introductory Methods of Numerical Analysis. Fifth. Vol. 4. PHI Learning Private Limited, New Delhi; 2012. 216–217 p). We have included this additional explanation in the revised manuscript.

1. We appreciate the reviewer’s insightful comment regarding the justification and validation of the numerical results presented in our manuscript.

The numerical results in this study are justified through a rigorous derivation of the corrected phase and geodesic equations based on well-established physical principles in geometrical optics. These corrected formulations are grounded in the standard optical path length concept and Fermat’s principle, as referenced in classical texts [1–3], ensuring their physical validity. Moreover, the application of Abelian U(1) gauge theory provides a theoretical framework that parallels established analogies in electrodynamics and field theory [5, 29].

To validate the relevance of our results, we have compared our numerical simulations with expected physical behavior in various media:

* In vacuum and homogeneous media, the simulation confirms that light propagates in straight lines and exhibits linear phase shifts, consistent with classical optics [1, 2].
* In anisotropic materials, the results show moderate phase fluctuations and curvature changes that align with known birefringence effects [10, 22].
* In optical metamaterials, we observe negative curvature and abrupt phase shifts, which are characteristic of negative-index materials, as documented in previous studies [20, 21].

These observations support the predictive capability of our reformulated approach and are further backed by foundational references in optics and gauge theory [1, 2, 5, 11, 29]. The approach offers a coherent link between geometric optics and gauge theory, enabling more accurate modeling of light behavior in complex optical systems.

We have clarified these points in the revised manuscript and provided additional citations where appropriate to reinforce the validation of our numerical results.

► Object: **Special Notes**

1. The manuscript contains incorrect grammatical sentences, such as misuse of passive and active voice. I encourage the authors to rewrite the whole manuscript and do rounds of proofreading to ensure the English is correct.

**Our response:**

We sincerely thank the reviewer for pointing out the grammatical issues and the misuse of passive and active voice in our previous manuscript. In response to this valuable feedback, we have carefully revised the entire manuscript with a focus on improving the grammar, sentence structure, and consistency in the use of voice. The revised version has undergone multiple rounds of proofreading to ensure that the English language is clear, correct, and appropriate for academic publication.

We hope that the current version meets the journal's standards and addresses the reviewer's concerns effectively.

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**Summary of changes:**

1. On pages 101 and 102, we replaced repeated instances of "author" in the manuscript with "In ref. [11], it used."
2. On page 104, we added an explanation regarding the applicability of Fermat's principle in light propagation.
3. On pages 105 and 106, we added explanations and illustrations concerning possible light paths (Figure 2) and curved light paths (Figure 3).
4. On pages 107 and 108, we added explanations and formulas related to the numerical method.
5. All equations are written straightforwardly.
6. Equation 37 is Equation 39 in the revised manuscript.
7. Equation 39 is Equation 41 in the revised manuscript.
8. Equation 40 is Equation 42 in the revised manuscript.
9. Equation 41 is Equation 45 in the revised manuscript.
10. Equation 42 is Equation 48 in the revised manuscript.