**Author responses to Reviewer**

We thank to the reviewer for the correction and several suggestion for our manuscript.

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| **No** | **Reviewer’s comments** | **Author responses** |
| E1 | Please make the motivation in the proper way, by not directly showing any party. Please note as well that in abstract is not common cite of the references. | In the manuscript, I have written my corrections in red ink. I have changed the title and rewritten the abstract. |
| E2 | As we mention in previous pre-review comment, The novelty of the manuscript is limited, as the proposed corrections largely revisit and refine existing formulations without introducing significant new physical insights or testable predictions.  To enhance the novelty and contribution of the manuscript, consider the following:   * Highlight the practical implications of the refined formulation, such as its relevance to experimental setups or its potential applications in anisotropic media or optical metamaterials. * Provide a comparative analysis of how the refined formulation improves upon the original in terms of mathematical consistency, physical predictions, or experimental feasibility.   **Otherwise, your refining idea should be commented on directly by the author whose article you argued.** | I have changed the title and rewritten the abstract. |
| E3 | The introduction lacks a clear articulation of the gaps in the previous study and how your work addresses these gaps. Explicitly stating the limitations of the prior formulation would help contextualize the significance of your corrections.  There is minimal discussion of why this reformulation is relevant or necessary. What are the broader implications of these corrections for physics, such as their potential impact on quantum field theory or optical engineering? | I have written my corrections in red ink. I rewrote paragraphs 2-4 and added paragraph 5 to clarify the gaps. |
| E4 | Just make it as citation.  Please revise this sentence Try this way: Telling what people did and what their gap and need to be improved. | I have revised the first sentence of the second paragraph. Additionally, I have rearranged the placement of sentences from the second and third paragraphs. |
| E5 | See above comment | I have written my corrections in red ink. |
| E6 | The corrections to the phase formula are heavily mathematical, and their physical interpretations are not always clear. For instance, how does the revised formulation impact the propagation of light in practical scenarios, such as in graded-index media or under weak-field approximations?  The methodology would benefit from a flowchart or schematic summarizing the correction process, linking each equation to the underlying physical concepts. | In the chapter of Correction of the Phase Formula added the first paragraph explaining the significance of the phase formula correction. Additionally, the steps for correcting the phase formula are shown in Figure 1.  In the chapter of Consequences of the CorrectionI added the first paragraph explaining the physical implications of the revised phase formula. In the last paragraph of the chapter, I added geodesic equation which used to develop the numerical simulation.  I added two subchapters, Optical Geodesic Equation and Simulation, to the chapter Consequences of the Correction.  In the subchapter SimulationI added several paragraphs which explain the simulation, simuation results, and numerical results. |