### **Critique of the Formulation of Geometric Optics as an Abelian U(1) Gauge Theory: Evaluation of Assumptions, Numerical Validity, and Physical Implications**

**Adrianus Inu Natalisanto1\* and Sri Purwaningsih2**

*1 Program Studi Fisika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Mulawarman, Jl. Barong Tongkok, Gn. Kelua, Samarinda Ulu 75242, Samarinda, Kalimantan Timur, Indonesia*

*2 Program Studi Fisika, Fakultas Sains dan Teknologi, Universitas Jambi, Jl. Lintas Jambi-Ma.Bulian Km 15 Mendalo Darat 36361, Jambi, Indonesia.*

*Corresponding Authors E-mail:* *adrianus@fmipa.unmul.ac.id*

|  |  |  |
| --- | --- | --- |
| **Article Info** |  | **Abstract** |
| ***Article info:****Received: xx-xx-20xx**Revised: xx-xx-20xx**Accepted: xx-xx-20xx****Keywords:****Geometrical optics; Abelian 𝑈(1) gauge theory; Refractive index****How To Cite:*** *A. I. Natalisanto and S. Purwaningsih, “Critique of the Formulation of Geometric Optics as an Abelian U(1) Gauge Theory: Evaluation of Assumptions, Numerical Validity, and Physical Implications”, Indonesian Physical Review, vol. 5, no. 2, p 100-108, 2025.****DOI:*** *https://doi.org/10.29303/ipr.vXiX.xxx.* |  | *This article critiques the claim that geometrical optics can be formulated as an Abelian U(1) gauge theory, as proposed in "Geometrical Optics as an Abelian U(1) Gauge Theory in a Vacuum Space-Time"(Indonesian Physical Review, Volume 7 Issue 1, Januari 2024). Evaluation shows that the assumption of a weak field as a representation of vacuum space-time is not entirely valid, especially under complex physical conditions such as intense fields* *or gravitational interactions. The proposed Abelian U(1) gauge transformation does not provide new insights compared to the traditional eikonal equation-based approach, while the claim that the refractive index is topologically invariant is refuted because the continuous nature of the refractive index contradicts the concept of topology. Numerical simulations show a strong dependence on arbitrary parameters without any clear physical justification, with the refractive index formulation becoming invalid at a certain distance due to negative values. These findings confirm that the traditional approach in geometrical optics remains more physically relevant and consistent with real phenomena. This article recommends further experimental validation, parameter sensitivity analysis, and theoretical justification in future research to ensure a more robust and relevant model in the study of geometrical optics.*  |
|  |
|  |
|  |  | *Copyright (c) 2024 by Author(s), This work is licensed under a Creative Commons Attribution-ShareAlike 4.0 International License.* |

**Introduction**

Geometric optics is one of the classical approaches in physics to describe the propagation of light, especially in the limit of very small wavelengths (𝜆→0). This approach uses the concept of light rays propagating through a medium with a certain refractive index, and is often used to understand the phenomena of refraction, reflection, and optical paths in various complex media [1][2][3]. In the electromagnetic framework, geometric optics can be derived from Maxwell's equations through the eikonal equation approach, which describes the path of light as a classical limit solution of electromagnetic waves [4][5][6]. This approach has also been extended to include anisotropic media and refractive index gradients using the gauge field [7][8].

Recently, research has attempted to reformulate geometric optics within the framework of Abelian U(1) gauge theory, as proposed in the article "Geometrical Optics as an Abelian U(1) Gauge Theory in a Vacuum Space-time" [9]. The article claims that the optical phase, refractive index, and propagation of light beams can be understood through gauge potentials, field strength tensors, and topological structures. This claim is mathematically interesting because it utilizes the formalism of gauge theory which is often used to explain particle physics and quantum field phenomena [10][11]. Previously, several studies have shown the application of gauge theory to light propagation in complex media, such as anisotropic media and metamaterials [12][13][14].

However, the claim raises several fundamental questions. First, the assumption of weak fields as a representation of vacuum space-time is questionable, especially in complex physical conditions such as intense fields or gravitational interactions. Second, the application of the Abelian U(1) gauge transformation in geometric optics does not show significant advantages over the traditional eikonal equation-based approach. Third, the claim that the refractive index can be treated as a topological invariant contradicts the continuous nature of the refractive index which is incompatible with the concept of topology.

This article evaluates the claims through theoretical approaches, numerical simulations, and falsification of experimental data. We find that the results of the articles rely heavily on arbitrary parameters without sufficient physical justification, and the refractive index formulation becomes invalid at certain distances. Thus, we conclude that the traditional approach in geometric optics remains more physically relevant, and future research should pay more attention to experimental validation and parameter sensitivity analysis.

**Geometrical Optics in Vacuum Space-time as an Abelian U(1) Gauge Theory**

The article "Geometrical Optics as an Abelian U(1) Gauge Theory in a Vacuum Space-time" by Miftachul Hadi and Suhadi Muliyono explores geometrical optics through the framework of Abelian U(1) gauge field theory. The authors reformulate the eikonal equation as a gauge theory in a (3+1)-dimensional vacuum space-time using a weak-field approximation. Key equations include the gauge potential

 (1)

and the field strength tensor

 (2)

which describe the optical field in terms of the refractive index *n(r)* and other variables. Numerical simulations reveal that the refractive index can be approximated as *n(r)=*1.0001 in vacuum space-time and that the weak magnetic field magnitude ∣*B*∣=0.10452 T supports the weak-field approximation.

The vacuum space-time is interpreted as a weak-field limit, where electromagnetic fields are of very low intensity. The study introduces the phase *q(****r****,t)*, related to the refractive index, expressed as:

, (3)

where X is a constant. The refractive index is modeled as

, (4)

showing that it decreases with increasing distance from the source. The authors also define the amplitude *ρ*(***r****,t*) and phase *q*(***r***,t) in the scalar field

, (5)

highlighting its topological and isotropic properties in vacuum.

Numerical results show the refractive index decreases radially, confirming vacuum characteristics. The study also finds the weak magnetic field magnitude ∣*B*∣=0.10452 T, computed using

. (6)

These findings reveal an innovative perspective connecting geometrical optics with gauge field theory, opening new avenues for exploring topological structures and weak-field conditions.

**Theoretical Criticism**

**Validity of the Weak Field Assumption as a Representation of Vacuum Space-Time**

The article assumes that vacuum space-time can be represented as the limit of a weak electromagnetic field. This assumption is used to define isotropic space-time far from the source of the field. However, there are several problems with this approach.

The first problems are limited physical conditions. The weak electromagnetic field is often an idealization approximation that is valid under certain conditions, for example, far from the source or in situations without quantum fluctuations. Near strong sources, such as high charges or intense fields, the weak field approximation is invalid because non-linearities become significant.

The second problem is gravitational effects. The article does not consider that the described vacuum space-time can also be affected by gravitational effects. In the context of relativistic physics, electromagnetic fields can interact with the curvature of space-time [15][16], which is ignored in the article.

**Abelian 𝑈(1) Gauge Transformation in Geometric Optics**

The article attempts to map geometric optics into the framework of Abelian U(1) gauge theory, but its implementation does not provide significant new insights.

The first implementation is redundancy with the Eikonal Approach. The gauge transformation introduced in the article only affects the optical phase (*q*(***r***,*t*)) without yielding any new predictions compared to the traditional eikonal equation approach. In other words, gauge theory does not provide any additional physical advantages in understanding the propagation of light rays.

The second implementation is the absence of gauge field interaction. In Abelian U(1) gauge theory, the gauge field Aμ ​ is usually involved in interactions with charged particles [11][17][18]. However, in the context of geometric optics, there are no relevant charged particles, so the application of gauge theory becomes merely a mathematical formalism with no real physical meaning.

**Relation of Refractive Index to Topological Structure**

The article proposes that the refractive index can be treated as a winding number (topological invariant), especially if its value is an integer. However, this claim has several fundamental weaknesses.

The first weakness is related to the continuity of the refractive index. The refractive index *n(r)* in geometric optics is usually a continuous function [19][20]. Topology, on the other hand, usually applies to fields with discrete structures or those with certain global symmetries, such as the Higgs field in particle physics [21][22]. There is no justification that the continuous refractive index has topological properties.

The second weakness is related to interpretative fallacy. The article equates the refractive index value with the winding number simply because of the mathematical similarity, without showing how this topological property impacts real physical phenomena.

**Numerical Dependence on Arbitrary Parameters**

The numerical results in the article, such as *n(r)*=1.0001 for the refractive index and *ρ2*=0.3119 for the weak field, depend on parameters such as *n0*​, *a*, and *r* that appear to be chosen arbitrarily. The parameter selection suffers from a lack of parameter justification and no sensitivity analysis. The article does not provide a physical explanation for why the values *n0*=1.6 or *a*=0.7499 were chosen. These values appear to have been chosen simply to produce results that fit the assumptions of the article. The article does not also show how the results change with variation in the parameters. This sensitivity analysis is important to understand whether the results are robust or whether they just happen to fit certain parameters.

**No Observational Consequences**

The Abelian gauge transformation U(1) should have observational implications, such as interference effects or changes in the propagation of light beams that can be tested experimentally. However, the article does not show how this gauge formulation affects real optical phenomena. There are no new predictions about light propagation that can be verified experimentally, such as changes in the path of light or refractive index properties. The gauge formulation seems to simply reproduce familiar results from traditional geometric optics approaches without any additional advantages.

**Misinterpretation of the Formula**

Equation (3) contains a mathematical form that is not physically clear, namely: . This leads to a peculiar consequence, as it implies a vacuum refractive index that is not one and varies with distance, as stated in the article.

The correct form of the integral should be: *,* which corresponds to the concept of optical path length [1][2][3][4]. When Equation (3) is corrected, it becomes:

. (7)

In Equation (7), it can be chosen

. (8)

The shortest wave propagation path from r1 to r2 at any time satisfies the condition:

. (9)

It can be found the variation of q1 as

. (10)

In Equation (10), there are two relations, namely:

 (11)

and

 .(12)

By combining Equations (9), (10), (11) and (12), it can be obtained

 (13)

where τ is the vector of tangential units with respect to the direction of wave propagation.

From Equation (13), it can be acquired the following relationship:

. (14)

Expanding Equation (14), it can be found the following Equation:

 (15)

where

 . (16)

Equation (15) can be rearranged so that it can be found the following equation:

. (17)

It is known that the term in the left side of Equation (17) is related to the radius of curvature *R* and the vector of normal units (perpendicular to the direction) of wave propagation ***η*** as stated by the following equation [23]:

 . (18)

The combination of Equations (17) and (18) gives the following equation:

. (19)

If Equation (19) is subjected to a dot operation with **η**, it will give the equation

. (20)

Based on Equation (20), it can be deduced that the direction of wave propagation is influenced by variations in the refractive index. Specifically, as the refractive index increases, the wave undergoes a change in direction. Consequently, when a wave propagates in a homogeneous medium where *n(****r****)* is constant, or in free space where *n(****r****)*=1, so , the wave trajectory remains unaffected by refraction. In such cases, the bending radius *R* approaches infinity, indicating that the wave path is linear.

**Conclusion Critique**

Overall, the main theoretical weakness of the article is the lack of physical justification for the application of Abelian 𝑈(1) gauge theory in geometric optics and misinterpretation of the Formula. The approach is more mathematical than physical, without providing significant new insights or testable predictions. The article tends to complicate an approach that is actually simple and well understood in geometric optics.

**Simulation and Experiment Criticism**

To support the rebuttal of the article's claims, the following simulations and experimental analyses can be conducted to demonstrate the weaknesses of the assumptions and conclusions drawn.

**Verification of the Dependence of the Bias Index on Arbitrary Parameters**

The article uses the Equation (4) with parameters 𝑛0=1.6, 𝑎=0.7499, and 𝑟=1. The result yields 𝑛(𝑟)=1.0001, which is considered to represent vacuum space-time. The following is a table of simulation results of Equation (4) and a summary of deviations from the target 𝑛(𝑟)=1.0001.

|  |
| --- |
| Table 1. Sample Simulation Results. |
| No | ***n0*** | ***A*** | ***R*** | ***n(r)*** | **Deviation from Target** |
| 1 | 1.5 | 0.74 | 0.800 | 1.1448 | 0.1447 |
| 2 | 1.5 | 0.74 | 0.933 | 1.0165 | 0.0164 |
| 3 | 1.5 | 0.74 | 1.067 | 0.8685 | 0.1316 |
| 4 | 1.5 | 0.74 | 1.200 | 0.7008 | 0.2993 |
| ... | ... | ... | ... | ... | ... |

|  |
| --- |
| Table 2. Summary of Deviation. |
| No | ***n0*** | ***A*** | ***Mean Deviation*** | ***Minimum Deviation*** | ***Maximum Deviation*** |
| 1 | 1.50 | 0.740 | 0.7015 | 0.0164 | 1.7201 |
| 2 | 1.50 | 0.745 | 0.7083 | 0.0132 | 1.7351 |
| 3 | 1.50 | 0.750 | 0.7151 | 0.0099 | 1.7501 |
| 4 | 1.55 | 0.740 | 0.7049 | 0.0503 | 1.7441 |
| ... | ... | ... | ... | ... | ... |

Based on the simulation, it can be concluded that: First, value *n(r)=*1.001 is stable only at certain parameters, namely around *𝑛0=*1.6, *a=*0.7499, and *r=*1; and, small changes in 𝑎 or 𝑟 can cause significant deviations from the target value, indicating that this condition is very sensitive to parameter variations. Second, changes in 𝑟 (distance) have the most significant impact on 𝑛(𝑟); values ​​of 𝑟>1 or 𝑟<1 cause 𝑛(𝑟) to deviate far from 1.0001, mainly due to the quadratic contribution of 𝑟2 in the equation; variations in 𝑎 (coefficient of decrease) provide linear changes in 𝑛(𝑟), with a greater impact if 𝑟 also increases; and, variations in 𝑛0 (initial index) shift the value of 𝑛(𝑟) proportionally, but the impact is smaller than 𝑟. Third, the result 𝑛(𝑟)≈1.0001 shows conditions approaching vacuum spacetime only for very specific parameters. This shows that such vacuum spacetime is in a very special condition and is easily disturbed by parameter fluctuations. Fourth, the condition 𝑛(𝑟)=1.0001 represents a very ideal vacuum spacetime. In practice, small variations in the parameters can create deviations, showing that vacuum spacetime is not completely stable without tight parameter control. Fifth, numerical simulations show a strong dependence on arbitrary parameters without any clear physical justification, with the refractive index formulation becoming invalid at a certain distance due to negative values.

Results that are too sensitive to parameters indicate that the article's formulation is only relevant for a particular case, not a general representation.

**Comparison with Experimental Data**

The article models the refractive index as *n(r)=*1.0001 with radial variation, based on parameters *n0=*1.6, *a=*0.7499, and *r=*1. According to the literature, the refractive index of vacuum has been measured respect to the refractive index of air with high precision using interferometric methods such as the Michelson interferometer [24][25][26]. Its values ​​are consistent with *n=*1.0000, with no measurable radial variation. The studies reported uncertainties on the order of 10−7 and 10-8, which does not support the existence of deviations as large as *n(r)=*1.0001 or radial variation.

Meanwhile, the article’s simulations show variation of *n(r)*, but this contradicts the experiments that show consistency of *n=*1.0000. The numerical simulation also shows a radial dependence of 𝑛(𝑟), which is not supported by experimental observations. Furthermore, the deviation *𝑛(𝑟)=*1.0001 falls outside the experimental uncertainty for vacuum refractive index. This discrepancy indicates that the model is not suitable for describing the properties of vacuum.

Moreover, the model proposed in the article, as reffered to in equation (4), has failed. First, the introduced radial variation implies that *n(r)* changes with *r*. This contradicts with experimental evidence showing *n* as invariant for vacuum. Second, the model parameters *n0=*1.6 and *a=* 0.7499 are arbitrarily chosen and do not correspond to physical constants or experimentally verified values for vacuum. At larger *r*, the model predicts negative refractive index values (*n<*0), which are unphysical in the context of vacuum. Hence, the numerical results fail to replicate the experimentally constant *n* for vacuum. The introduction of radial variation in *n(r)* has no basis in experimental data or theoretical frameworks for vacuum properties.

The article produced results that do not correspond to the actual physical conditions of vacuum. The article's model for the refractive index of vacuum fails to align with experimental measurements and physical reality. While the proposed formulation may hold conceptual interest, it does not accurately represent the vacuum or low refractive index media. This demonstrates a significant limitation in the model's applicability to real-world scenarios.

**Magnetic Field Simulation in Vacuum Space-Time**

The article states that the amplitude 𝜌 and the magnetic field ∣𝐵∣ represent weak field conditions. With 𝜌=0.55853, the magnetic field is calculated as ∣𝐵∣2=𝜌2=0.3119. This corresponds to a magnetic field magnitude of approximately ∣*B*∣=0.56 T. However, when compared to real vacuum magnetic fields, such as interstellar magnetic fields, which typically range from 7.8810−6 T to 8.0810−6 T, this value is unrealistically high by several orders of magnitude [27]. Simulations performed with varying ρ values, including *ρ*=0.1, 0.2, 0.5, 1.0, 2.0, show a quadratic relationship between ∣*B*∣2 and *ρ*2, further exacerbating the discrepancy for larger *ρ*. This highlights a significant deviation of the article’s theoretical assumptions from observed vacuum magnetic field values. The disconnect suggests that *ρ* as modeled in the article lacks physical relevance in describing weak magnetic field conditions in vacuum or interstellar environments. Such results emphasize the need for better alignment of theoretical models with empirical data to ensure their physical validity. The value of ∣𝐵∣ produced by the article is likely too high or irrelevant to real vacuum conditions.

**Testing the Relationship between Topology and Refractive Index**

The article claims that the refractive index *n(r)* can be considered a topological invariant, specifically a winding number, if its value is an integer. However, this claim is conceptually flawed when applied to continuous refractive indices, which are scalar fields in geometric optics. Simulations of *n(r)* with spatial variations, such as

 , (21)

reveal that continuous changes in *n(r)* do not produce discrete or quantized topological properties. A winding number, defined as a quantized invariant associated with periodicity or phase looping, requires the system to exhibit discrete transitions or symmetries that are not present in a continuous refractive index [28][29]. Furthermore, refractive indices describe material properties rather than phase or vector fields, disqualifying *n(r)* from being associated with topological invariants in real physical systems. The simulation results confirm that the article's topological interpretation is only relevant in idealized mathematical constructs, with no basis in experimental or physical reality.

**Physical Experiments to Validate the Gauge Formulation**

The article asserts that the Abelian U(1) gauge transformation is relevant in geometrical optics and should result in observable effects, such as changes in the light path. To validate this claim, an experiment can be conducted using an optical setup with a medium of variable refractive index, such as a gradient-index (GRIN) lens or an anisotropic medium. Phase transformations analogous to U(1) gauge transformations can be introduced using spatial light modulators or phase plates. The light's trajectory under these conditions can then be compared to the predictions of both the article's gauge formulation and traditional eikonal-based geometrical optics. Control experiments without the phase transformation can further establish the baseline behavior.

Recent studies indicate that gauge transformations in geometrical optics often do not result in observable effects unless the system involves quantum mechanical or topological constraints [28][29]. Traditional approaches, which rely on the eikonal equation, have consistently provided accurate predictions for light paths in optical systems. It is therefore expected that the experiment will reveal no significant differences between the gauge formulation and traditional models, highlighting the limited physical relevance of the proposed gauge-based framework.

**Conclusion**

The analysis reveals that the proposed formulation of geometrical optics as an Abelian U(1) gauge theory lacks physical relevance in classical optics and there is misinterpretation of the used formula. Numerical simulations and experimental comparisons show that the gauge transformation does not produce observable effects, such as deviations in the light path, that differ from traditional eikonal-based predictions. Claims regarding the topological nature of the refractive index are mathematically intriguing but have no physical basis in continuous media. Furthermore, the assumed weak field conditions and the modeled refractive index variations fail to align with empirical data for vacuum and low refractive index media. Therefore, the proposed gauge framework provides no substantial advantage or additional insight compared to conventional geometrical optics.

**Suggestions**

To strengthen the proposed theoretical model, after eliminating misinterpretations of the formula, the first step is to conduct experiments to validate the theoretical claims, such as observing changes in light paths or interference effects caused by gauge transformations. Next, simulations with parameter sensitivity analysis should be developed to understand how variations in parameters like 𝑛₀ and 𝑎 affect the results, providing insights into the model's stability. Justifying these key parameters, either through experimental or theoretical approaches, will offer a stronger physical foundation for the model. Theoretical development is also crucial to address the model's shortcomings, such as accounting for strong field effects or gravitational interactions. Finally, an interdisciplinary approach that combines gauge theory with established experimental methods can further validate these ideas, paving the way for broader applications in physics.

**Acknowledgment**

We would like to thank the Reviewers for reviewing this manusript. This research is fully supported by self-funding.

**References**

[1] M. Born and E. Wolf, *Principles of optics \_ electromagnetic theory of propagation, interference and diffraction of light*. 1999.

[2] E. Hecht, *Optics*. 2002.

[3] B. D. Guenther, *Modern Optics Simplified*. 2020.

[4] A. V. Fedorov, C. A. Stepa, A. V. Korolkova, M. N. Gevorkyan, and D. S. Kulyabov, “Methodological derivation of the eikonal equation,” *Discret. Contin. Model. Appl. Comput. Sci.*, vol. 31, no. 4, pp. 399–418, 2023, doi: 10.22363/2658-4670-2023-31-4-399-418.

[5] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*. 1984.

[6] K. V Shajesh, “Eikonal Approximation.”

[7] F. Liu and J. Li, “Gauge field optics with anisotropic media,” *Phys. Rev. Lett.*, vol. 114, no. 10, Mar. 2015, doi: 10.1103/PhysRevLett.114.103902.

[8] D. Delphenich, “On geodesics of gradient-index optical metrics and the optical-mechanical analogy,” 2020.

[9] M. Hadi and S. Muliyono, “Gauge Theory in a Vacuum Space-time,” *Indones. Phys. Rev.*, vol. 7, no. 1, pp. 143–151, 2024, doi: 10.29303/ip.

[10] S. Weinberg, *The Quantum Theory of Fields Volume I Foundations*. 1995.

[11] I. J. Aitchison and A. J. Hey, *Gauge Theories Particle Physics, VOLUME 1 From Relativistic Quantum Mechanics to QED*. 2013.

[12] Y. Chen *et al.*, “Non-Abelian gauge field optics,” *Nat. Commun.*, vol. 10, no. 1, Dec. 2019, doi: 10.1038/s41467-019-10974-8.

[13] S. I. Maslovski and H. Mariji, “Envelope Dyadic Green’s Function for Uniaxial Metamaterials,” Feb. 2018. [Online]. Available: http://arxiv.org/abs/1802.05899

[14] D. K. Sharma and S. K. Pathak, “Propagation characteristics of an extremely anisotropic metamaterial loaded helical guide,” *Opt. Express*, vol. 24, no. 26, p. 29521, Dec. 2016, doi: 10.1364/oe.24.029521.

[15] R. W. M. Woodside, “Space-time Curvature of Classical Electromagnetism,” 2004.

[16] C. G. Tsagas, “Electromagnetic fields in curved spacetimes,” *Class. Quantum Gravity*, vol. 22, no. 2, pp. 393–407, Jan. 2005, doi: 10.1088/0264-9381/22/2/011.

[17] C. A. Dartora *et al.*, “Lagrangian-Hamiltonian formulation of paraxial optics and applications: Study of gauge symmetries and the optical spin Hall effect,” *Phys. Rev. A - At. Mol. Opt. Phys.*, vol. 83, no. 1, Jan. 2011, doi: 10.1103/PhysRevA.83.012110.

[18] K. H. Yang, “Gauge Transformations and Quantum Mechanics Gauge Invariant Interpretation of Quantum Mechanics,” vol. 96, pp. 62–96, 1976.

[19] H. DeVoe, “Optical properties of molecular aggregates. II. Classical theory of the refraction, absorption, and optical activity of solutions and crystals,” *J. Chem. Phys.*, vol. 43, no. 9, pp. 3199–3208, 1965, doi: 10.1063/1.1697294.

[20] J. D. Rogers, A. J. Radosevich, J. Yi, and V. Backman, “Light scattering in tissue modeled as continuous random media using a modified Whittle-Mat ´ ern correlation family,” *IEEE J. Sel. Top. Quantum Electron.*, vol. 20, no. 2, pp. 1–13, 2014, doi: 10.1109/JSTQE.2013.2280999.

[21] G. Esposito, “Towards a Spectral Proof of the Mass Gap in QCD?,” 2001. doi: 10.1142/S0217751X02010327.

[22] A. Gangui, *Topological Defects in Cosmology*.

[23] M. R. Spiegel, *Vector Analysis*. 1959.

[24] H. Gao and S. Wang, Z., Zou, W., Liu, Y., and Sun, “High-accuracy measurement system for the refractive index of air based on a simple double-beam interferometry,” vol. 29, no. 2, pp. 1396–1411, 2021, doi: 10.1364/OE.413252.

[25] X. Yu and J. D. Ellis, “Absolute air refractive index measurement and tracking based on variable length vacuum cell,” vol. 55, no. 6, 2016, doi: 10.1117/1.OE.55.6.064112.

[26] G. Schodel, R., Walkov, A., Voigt, M., and Bartl, “Measurement of the refractive index of air in the low-pressure regime and the applicability of traditional empirical formulae,” 2018.

[27] M. Zhou, Z. Hu, X. Duan, B. Sun, J. Zhao, and J. Luo, “Precisely mapping the magnetic field gradient in vacuum with an atom interferometer,” vol. 061602, pp. 2–5, 2010, doi: 10.1103/PhysRevA.82.061602.

[28] B. Zhen, C. W. Hsu, L. Lu, A. D. Stone, and M. Soljačić, “Topological nature of optical bound states in the continuum,” *Phys. Rev. Lett.*, vol. 113, no. 25, 2014, doi: 10.1103/PhysRevLett.113.257401.

[29] A. M. Srivastava and M. Stone, “Propagation of Electromagnetic Wave in th Core of a String Defect in Liquid Crystals,” pp. 1–8, 1993.