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Analytical Approach of Long Waves Dynamics in an Estuary (Case Study in Karang Mumus River Estuary)

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Abstract. A mathematical model is developed for describing a propagating long wave in an estuary. Wave dynamics in an estuary will employ the Saint Venant equation in the form of a nonlinear partial differential equation (PDE), which consists of equations of mass conservation and momentum conservation. Waves propagate in an estuary usually generated by tides entering a gentle slope channel. The analytical approach is used to solve this non-linear PDE approach using a perturbation method. This paper considers only a zero order. A cross-differentiation between conservation of mass and conservation of momentum equations result in the Bessel differential equation. The solution of zero order gives decreasing amplitude of the long wave to the end of the estuary.

1. Introduction

An estuary is semi-closed water that is connected to the sea so that it is strongly influenced by freshwater from rivers and sea water with high salinity coming from the sea. The combination of seawater and freshwater will produce a brackish water as a distinct area with varied environmental conditions. This area can also be said to be a very dynamic area because there are always processes and changes in physical, chemical and biological environments [1]. The study of waves in the estuary is very important, one example of those estuaries is that occur along the Mahakam River estuary, in Samarinda City, East Kalimantan. A mixing between freshwater and seawater in the estuary is a factor that increases water fertility and makes the estuary is one of the most productive natural habitats [2]. The Mahakam River always supports life surrounding communities mainly as source of water, fisheries potential and as transportation infrastructure. The mix of seawater with freshwater makes the Mahakam estuary area is multi-purposes utilization, and has a fluctuating water level due to tidal dynamics that is interested to be studied.

Rainwater that flows into the river then meets sea water is a factor in the formation of the estuary. Other factors that influence the formation of estuaries are the occurrence of tides caused by interactions in the planetary system, especially between the earth, the sun, and the moon periodically depending mainly on the frequency of earth's rotation, earth's orbit, and the moon resulted in long period of waves [3]. The change in salinity in the estuary is strongly influenced by the tide. The existence of freshwater flow that occurs continuously from the upstream of the river and the process of water movement due to tidal currents that transport minerals, organic material, and sediments are the



basic ingredients that can support the productivity of the waters in the estuary area which exceeds the productivity of the high seas and freshwater waters. The theory of tidal hydraulics is specifically related to equations that describe the relationship between the shape of the estuary and its hydrology. As with all open channels, tidal hydraulics in the estuary can be explained by the Saint Venant equation which is one of the mathematical models that are often used to present fluid flow in open channels. The Saint Venant equation is a governing equation consisting of conservations of mass and momentum equations in the form of partial differential equation (PDE). The equation of momentum conservation involves several external forces such as gravity, bottom topography, and bottom roughness of the channel.

Efforts to find an analytic solution from the equation of viscosity flow began to be carried out by Berre de Saint Venant in 1843. Analytical solutions are one option to change PDEs that are easy to apply. In river systems, general equations can be assumed to have constant cross-sections. Deep understanding to the dynamics might be made by solving Saint Venant equations for estuaries with varied cross-sections [4]. However, a linearized equation with linear and horizontal friction can be used. Analytical solutions that are currently being developed, especially using one-dimensional models, require some assumptions that bottom friction is homogeneous along the estuary, and no lateral input of discharge. Such general assumptions are also used by Savenije [5]. Based on this quasi-nonlinear approach, Savenije presents a completely explicit solution from the tidal dynamics in an estuary [6]. An analytical approach was also taken by [7] to predict freshwater discharge in estuaries based on observations of tidal levels. With the same model, explicit solutions are also presented and applied in the Sebou Morocco river estuary [8].

In this study a different method of solving non-linear partial differential equations by using the perturbation method to find a linear component of PDE, and transformation method to find a similarity to the shape of special functions. One of the analyses utilizing a perturbation method is to find the asymptotic expansion. Mathematically, this method was performed to get a solution approach to the equations which are not easy to obtain [6]. The purpose of this study was to solve the Saint Venant equation for the tidal propagation approximated by the perturbation method. While the benefits of this study are to estimate the potency of brackish water that is well used for the benefit of aquaculture activities in fish ponds close to the estuary. Logically, fish ponds near a river mouth will have abundant sea water compared to the far one. Eventually, a good farm has to have a mixing pond that produces brackish water with required special salinity to grow economical commodities.

2. Mathematical model

Conceptual sketches of the present model of long wave propagation generated by tidal force in an estuary is presented in **Figure 1**, x is an axis of propagation as longitudinal coordinates measured from the river mouth, h is the depth of an estuary, z is level of water fluctuation, z_0 is bottom elevation, h_0 is the mean water depth at the river mouth, HW is level of high water and LW is level of low water. A sketch of the waves in the estuary area is after [3] as follow.

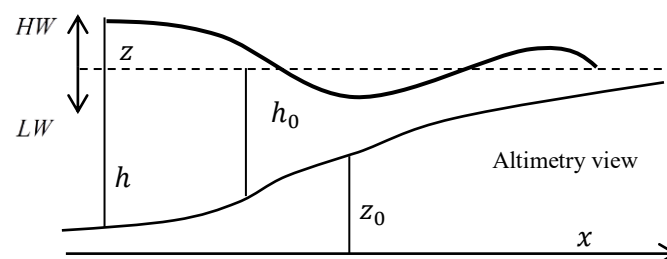


Figure 1. Sketches of Waves in an Estuary.

The equation used in this study is a one-dimensional Saint Venant equation, assuming that long waves are generated by tides, flowing into a constant channel width with the varied depth to be smaller

than the width, no freshwater discharge, compared to tidal discharge, and Froude number is set to be about two. Some parameters are used as follow: a tidal period T is set to be the diurnal tide period, a velocity V , η is a fluctuating deviation from mean water level, ρ as the water density is set to be homogenous, g is the acceleration due to gravity, C_h is Chezy's friction factor as external force from the bottom, A is the cross-sectional area of the considered channel, and $Q = AV$ as the tidal flow discharge.

The governing equation consists of conservation of mass and momentum equations. These two equations are the main equations of the Saint Venant for open channels to explain the tidal dynamics in the estuary in one-dimensional equations as follows

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad (1)$$

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial z_b}{\partial x} + g \frac{\partial h}{\partial x} - g \frac{h}{2\rho} \frac{\partial \rho}{\partial x} + g \frac{V|V|}{C_h^2 h} = 0. \quad (2)$$

3. Scaling the equations

A dimensionless process is an important physical process to guarantee that to develop a model will has no a dimension problem during a solving process afterward. If a model is assumed to have a length variable x then the dimensionless length variable is x^* can be stated following [9], where $[x]$ is relevant length

$$x = [x]x^*. \quad (3)$$

In this section dimensionless parameters can be identified which can be used to describe the characteristics of tidal waves that occur in the river estuary channel by taking certain constant values, where the corresponding reference value of the main quantity is adopted as a constant scale. The dimensionless process is stated in **Table 1** as follows.

Table 1 Non-dimensional processes.

Dimensional variables	Dimensionless variables
$x = h_0 x^*$	x^*
$t = t^* / \omega$	t^*
$V = V^* \sqrt{gh_0}$	V^*
$h = h_0 h^*$	h^*
$h = h_0 - x \tan \theta + \eta$	$h^* = 1 - x^* S + \eta^*$

This dimensionless form is needed to compare the terms in the equation when looking for a solution using a perturbation method. With this dimensionless form it can also be known which component has more significant value in line with solving process, where each subsequent solution will maintain to be closer to the exact solution than the solution previously obtained.

The dimensionless process of equations (1) and (2) give the following results

$$\omega \frac{\partial h^*}{\partial t^*} + \frac{\sqrt{g}}{\sqrt{h_0}} \frac{\partial(h^* V^*)}{\partial x^*} = 0, \quad (4)$$

$$\omega \frac{\partial V^*}{\partial t^*} + \frac{\sqrt{g}}{\sqrt{h_0}} V^* \frac{\partial V^*}{\partial x^*} + \frac{\sqrt{g}}{\sqrt{h_0}} \frac{\partial h^*}{\partial x^*} + \frac{\sqrt{g}}{\sqrt{h_0}} (S - S_\rho + S_f) = 0. \quad (5)$$

4. The perturbation method

The perturbation theory in mathematics can be used to get a solution approach to complex differential equations and when the exact solution is hard to find. In general, small disturbances in the physical system are denoted by ε , where ε is a perturbation parameter that has very little value or can be lesser than one. The way to get a solution approach to the perturbation method is to use matched asymptotic expansion. The asymptotic expansion method is defined as follow

Definition-1

Function $\phi_1(\varepsilon), \phi_2(\varepsilon), \dots$ forms an asymptotic row for $\varepsilon \rightarrow \varepsilon_0$ if and if only $\phi_{m+1} = O(\phi_m)$ for $\varepsilon \rightarrow \varepsilon_0$ for every m

Definition-2

If $\phi_1(\varepsilon), \phi_2(\varepsilon), \dots$ is asymptotic row, then $f(\varepsilon)$ is the asymptotic expansion for n-th term, if and if only

$$f = \sum_{k=1}^m a_k \phi_k + O(\phi_m), \text{ for } m = 1, 2, \dots, n; \text{ and } \varepsilon \rightarrow \varepsilon_0,$$

Coefficient a_k does not depend on ε , consequently it can be written as

$$f \approx a_1 \phi_1(\varepsilon) + \dots + a_n \phi_n(\varepsilon) \text{ for } \varepsilon \rightarrow \varepsilon_0. \quad (6)$$

Functions ϕ_k are said as scale or gauge function [10]. Gauge function is positive, monotone function and stands on an interval where $0 < \varepsilon < \varepsilon_0$. The simplest form of gauge function and the most used is the power of ε , namely $1, \varepsilon, \varepsilon^2, \varepsilon^3, \dots$.

5. Solution of long wave equation

The conservation of mass conservation equation and momentum conservation equations in equations (3) and (4) can be uniformly written as

$$a \frac{\partial h^*}{\partial t^*} + \frac{\partial(h^* V^*)}{\partial x^*} = 0, \quad (7)$$

$$a \frac{\partial V^*}{\partial t^*} + V^* \frac{\partial V^*}{\partial x^*} + \frac{\partial h^*}{\partial x^*} + (S + S_f) = 0. \quad (8)$$

The value of $a = \omega \sqrt{h_0 / g}$ is constant. Both equations (7) and (8) will be modified to obtain a long wave equation. The first step is to multiply V^* on the equation (7) and multiply h^* on the equation obtained (8)

$$a V^* \frac{\partial h^*}{\partial t^*} + V^* \frac{\partial(h^* V^*)}{\partial x^*} = 0, \quad (9)$$

$$ah^* \frac{\partial V^*}{\partial t^*} + h^* V^* \frac{\partial V^*}{\partial x^*} + h^* \frac{\partial h^*}{\partial x^*} + h^* (S + S_f) = 0. \quad (10)$$

Add these two equations above, yield

$$a \left(\frac{\partial}{\partial t^*} (h^* V^*) \right) + \frac{\partial}{\partial x^*} (h^* V^{*2}) + h^* \frac{\partial h^*}{\partial x^*} + h^* (S + S_f) = 0. \quad (11)$$

Differentiating equation (7) with respect to t^* and equation (11) with respect to x^* , so it can be resulted a new equation

$$a^2 \frac{\partial^2 h^*}{\partial t^{*2}} + a \frac{\partial^2 (h^* V^*)}{\partial x^* \partial t^*} = 0, \quad (12)$$

$$a \frac{\partial^2 (h^* V^*)}{\partial x^* \partial t^*} + \frac{\partial^2}{\partial x^{*2}} (h^* V^{*2}) + \frac{\partial h^*}{\partial x^*} \frac{\partial h^*}{\partial x^*} + h^* \frac{\partial^2 h^*}{\partial x^{*2}} + (S + S_f) \frac{\partial h^*}{\partial x^*} = 0. \quad (13)$$

By cross-differentiation of equations (7) and (11), equations (12) and (13) yield

$$a^2 \frac{\partial^2 h^*}{\partial t^{*2}} = \frac{\partial^2}{\partial x^{*2}} (h^* V^{*2}) + \frac{\partial h^*}{\partial x^*} \frac{\partial h^*}{\partial x^*} + h^* \frac{\partial^2 h^*}{\partial x^{*2}} + (S + S_f) \frac{\partial h^*}{\partial x^*}. \quad (14)$$

Because $h^* = (1 - x^* S + \varepsilon \eta^*)$ and $V^{*2} = (1 - x^* S + \varepsilon \eta^*)$ in which $\varepsilon < 1$, so the equation (14) can be written as

$$a^2 \frac{\partial^2 \eta^*}{\partial t^{*2}} = 3(1 - x^* S) \frac{\partial^2 \eta^*}{\partial x^{*2}} + (S_f - 5S) \frac{\partial \eta^*}{\partial x^*} + 3\varepsilon \eta^* \frac{\partial^2 \eta^*}{\partial x^{*2}} + 3\varepsilon \left(\frac{\partial \eta^*}{\partial x^*} \right)^2 + \frac{(SS_f + 2S^2)}{\varepsilon}. \quad (15)$$

The next step is to use an assumption related to the form of asymptotic expansion. Here, the asymptotic expansion is written in the form of

$$\eta^*(x^*, t^*) = \varepsilon^0 \eta_0^* + \varepsilon^1 \eta_1^* + \varepsilon^2 \eta_2^* + \cdots + \varepsilon^n \eta_n^*. \quad (16)$$

Where $\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots, \varepsilon^n$ are in the row of gauge function $\eta^*(x^*, t^*)$ is function of x^* and t^* variables that state solution for zero order, first order to n -order. Substitution equations (16) to (15), we obtain zero order of linear equation $O(1)$ as follows

$$a^2 \frac{\partial^2 \eta_0^*}{\partial t^{*2}} - 3(1 - x^* S) \frac{\partial^2 \eta_0^*}{\partial x^{*2}} - (S_f - 5S) \frac{\partial \eta_0^*}{\partial x^*} = 0. \quad (17)$$

Solution for η_0^* is depending on variable t^* that it can be written as function of

$$\sim e^{it^*}. \quad (18)$$

By substituting second differential of equation (18) into equation (17) we can obtain

$$3(1-x^*S)\eta_{0x^*}^* + (S_f - 5S)\eta_{0x^*}^* + a^2\eta_0^* = 0. \quad (19)$$

Performing variable transformation in the form of $z^2 = (1 - x^*S)$ for the equation (19) we will have

$$z^2 \frac{\partial^2 \eta_0^*}{\partial z^2} + \left(\frac{2}{3} - \frac{S_f}{3S} \right) z \frac{\partial \eta_0^*}{\partial z} + \frac{a^2}{3S^2} z^2 \eta_0^* = 0. \quad (20)$$

Based on the formula developed by [11], equation (20) can be solved by transforming into Bessel differential as follows

$$z^2 \frac{d^2 \eta_0^*}{dz^2} + (2k+1)z \frac{d\eta_0^*}{dz} + (\alpha^2 z^{2r} + \beta^2) \eta_0^* = 0. \quad (21)$$

Where k, α, r , and β are constants that have values related to our problem of the estuary in (20). The solution of (20) in Bessel function will be

$$\eta_0^* = z^{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left[c_1 J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} z \right) \right] \quad (22)$$

or

$$\eta_0^* = \sqrt{(1 - Sx^*)}^{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left[c_1 J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} \sqrt{(1 - Sx^*)} \right) \right]. \quad (23)$$

Initial condition for dimensionless deviation from mean water level at the mouth of estuary is set to be $\eta_0^*(0) = 0.155$ and $\eta_0^*(\infty) = 0$ [12]. By entering these boundary conditions, the solution is

$$\eta_0^* = \frac{0.155}{J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} \right)} \left[\sqrt{(1 - Sx^*)}^{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} \sqrt{(1 - Sx^*)} \right) \right]. \quad (24)$$

Then the fluctuating deviation of dimensional form can be written as follows

$$\eta(x, t) = h_0 \frac{0.155}{J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} \right)} \left[\sqrt{(1 - Sx^*)}^{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} J_{\left(\frac{1}{6} + \frac{S_f}{6S}\right)} \left(\frac{a}{\sqrt{3S}} \sqrt{(1 - Sx^*)} \right) \right] e^{i\omega t}. \quad (25)$$

This paper is also reviewed through the completion of graph visualization, based on possible data in the Karang Mumus River estuary, for example we set the parameters as follows: $S = 0.0001$ and 0.001 , $S_f = 0.003902$, $h_0 = 10$ m, $T = 6$ hours, $g = 9.8$ m/s², and $x = 10000$ m and 1000 m. **Figure 2** and **Figure 3** shows the results of our work eventually, the propagation of long waves in to an estuary.

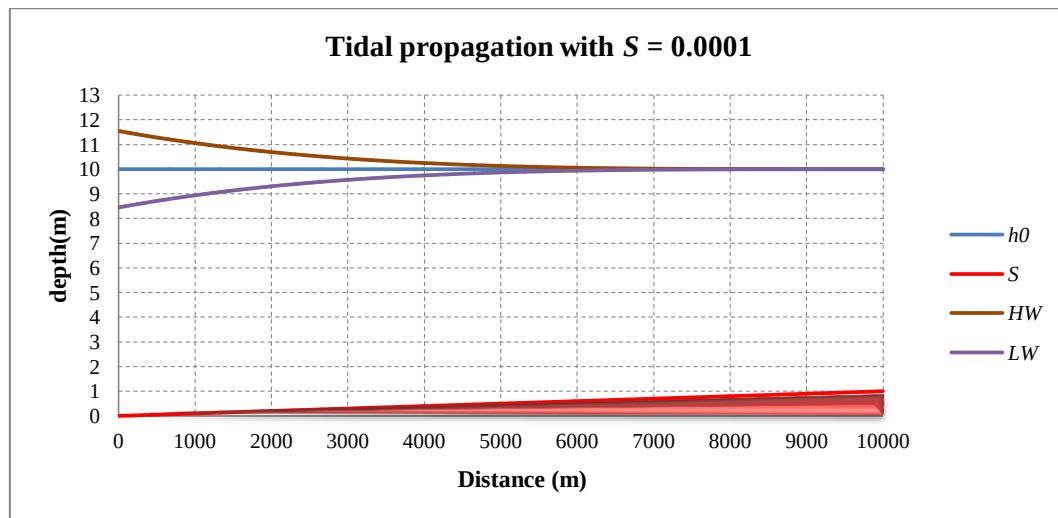


Figure 2. Curve $h(x)$ to x in the tidal propagation with $S = 0.0001$. HW at the mouth of the estuary is 1.55 meters and LW at the mouth of the estuary is -1.55 meters from the mean sea level of h_0 . At a distance of 6000 meters, the tides no longer exist.

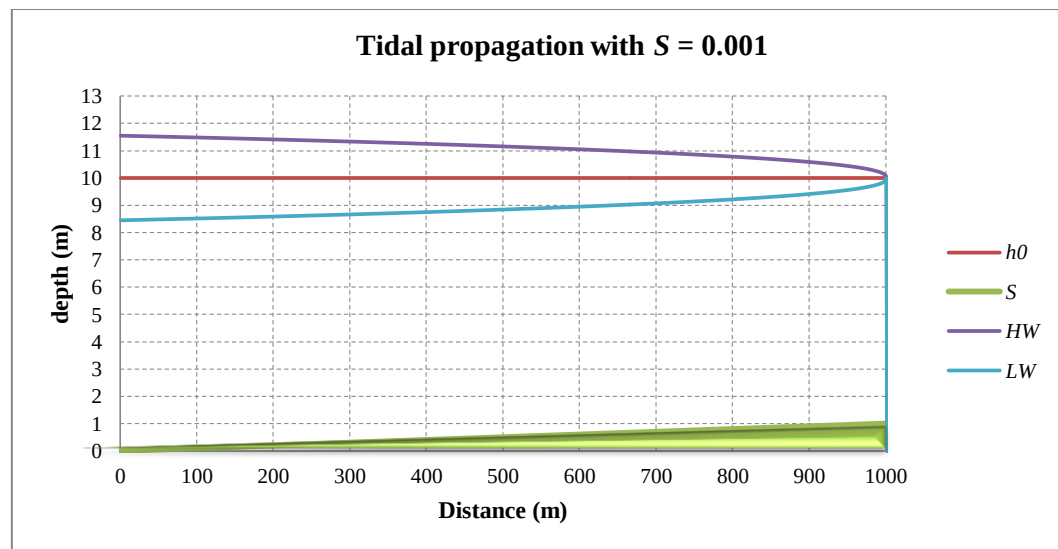


Figure 3. Curve $h(x)$ to x in the tidal propagation with $S = 0.001$. HW at the mouth of the estuary is 1.55 meters and LW at the mouth of the estuary is -1.55 meters from the mean sea level at h_0 . At a distance of 1000 meters, the tides no longer exist.

6. Conclusion

Non-linear governing equations of Saint Venant in one-dimension have been solved using perturbation methods that lead to PDE of long wave equation. A cross-differentiation between mass and momentum equations resulted in the Bessel differential equation. The solution is also reviewed through the completion of graph visualization. The solution of zero order gives decreasing amplitude of the long wave to the end of the estuary as we expected beforehand. The deviation of water level

from mean water level gives the potency of brackish water to be benefited to develop aquaculture activities.

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