Original Article

Development of nonparametric geographically weighted regression using truncated spline approach

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Abstract

Nonparametric geographically weighted regression with truncated spline approach is a new method of statistical science. It is used to solve the problems of regression analysis of spatial data if the regression curve is unknown. This method is the development of nonparametric regression with truncated spline function approach to the analysis of spatial data. Spline truncated approach can be a solution for solving the modeling problem of spatial data analysis if the data pattern between the response and the predictor variables is unknown or regression curve is not known. This study focused on finding the estimators of the model nonparametric geographically weighted regression by maximum likelihood estimator (MLE) and then these estimators are investigated the unbiased property. The results showed nonparametric geographically weighted regression with truncated spline approach can be used in spatial data to solve problems regression curve that cannot be identified.

Keywords: nonparametric geographically weighted regression, truncated spline, spatial data, unbiased estimation

1. Introduction

The development in science and technology has been rapid over the years, and the observation of the signs in nature led to unusual patterns, therefore it is difficult to predict the behavior of nature. In the past decade, the start and end of dry and rainy season in various geographical areas could easily be predicted, hence farmers were able to prepare when to harvest and plant rice; however in the recent years it has become difficult. Examples of events that shape the unusual and irregular patterns of nature are the issue of the percentage of poverty, underdevelopment, literacy rates, and ignorance that is growing and uneven development in each area along with other variables.

For more than a century, geographers, economists, city planners, business strategy experts, regional scientists and other social scientists have tried to explain the "why" and "where" of implemented activities. This encourages the proliferation of research on the influence of spatial effect which is able to explain the effect of events caused by their geographical influence. Some methods of spatial analysis that has been developed are geographically weighted regression or known as geographically weighted regression (GWR).

GWR was first introduced by Fotheringham in 1967. The response variables in GWR model contains predictor variables that each regression coefficient depends on the location where the data is observed. The development of researches on GWR was as follows. A new model of GWR

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which is geographically weighted Poisson regression models was found (Nakaya *et al.*, 2005). A model that incorporates regression model global and GWR, known as mixed geographically weighted regression models, was produced (Chang-Lin *et al.*, 2006). Another GWR model known as spatio-temporal (Demsar *et al.*, 2008), was also developed in the field of spatially related time series. Furthermore, a research on geographically and temporally weighted regression model (Huang *et al.*, 2010) and research on spatial panel data with geographically weighted regression panel method (Yu, 2010) were examined multivariate geographically Weighted regression (MGWR). This study discusses the estimation of the model using weighted maximum likelihood method (Harini *et al.*, 2010), geographically and temporally weighted likelihood regression model (Wrenn & Sam, 2014), and geographically weighted regression with spline approach (Sifriyani *et al.*, 2017).

The methods which were developed in this study are still in the form of linear and assumed that the data pattern is known. In fact, when modeling the data the question is, it true that all relationships between predictor variables and the response variables form a known regression curve. In reality, of course not all the data pattern of relationships has a known regression curve so it is necessary to use nonparametric regression analysis for solving this problem. A good model should be viewed from various aspects and put a proper modeling issues in its portion. The difference between the environmental characteristics and geographic location of observation, effects the observations to have different variations or there are different influences of predictor variables on the response variables for each observation location. How to solve when predictor variables have variable patterns which are not following a specific pattern on response variables and regression curve is not known. In this case, the GWR models have not been able to solve this problem so the authors developed a nonparametric regression model that is GWR otherwise known as nonparametric geographically weighted regression.

This paper is organized as follows; Section 1: Introduction, Section 2: Model Nonparametric Geographically Weighted Regression Using Truncated Spline Approach, Section 3: Result and Discussion and Section 4: Conclusion.

2. Nonparametric Geographically Weighted Regression Models Using Truncated Spline Approach

Nonparametric geographically weighted regression models with truncated spline approach are the development of nonparametric regression for spatial data where the parameter estimator is local to each observation location. Spline trun-cated approach used to solve the problems of spatial analysis regression curve is unknown. In the regression model assump-tions used are normally distributed error with zero mean and variance $\sigma^2(u_i, v_i)$ at each location (u_i, v_i). Location coor-dinates (u_i, v_i) is one of the important factors in determining the weighting used for estimating the parameters of the model. Given the data $(x_{1i}, x_{2i}, \ldots, x_{li}, y_i)$ and the relationship bet-ween $(x_{1i}, x_{2i}, \ldots, x_{li},)$ and y_i assumed to follow a nonpara-metric regression model as follows:

$$y_i = f(x_{1i}, x_{2i}, \ldots, x_{li},) + \varepsilon_i, \quad i = 1, 2, \ldots, n \tag{1}$$

with y_i as the response variable and $f(x_{1i}, x_{2i}, \ldots, x_{li},)$ is a function of the unknown regression curve shape which is assumed to be additive. If $f(x_{1i}, x_{2i}, \ldots, x_{li},)$ is approached by a multivariable spline function, it can be written as follows:

$$y_i = \sum_{p=1}^{l} f_p(x_{pi}) + \varepsilon_i$$

Given $f_p(x_{pi}) = f$ then f approximated by a truncated spline functions for each location (u_i, v_i), defined as follows:

$$= \beta_0(u_i, v_i) + \sum_{p=1}^{l} \sum_{k=1}^{m} \beta_{pk}(u_i, v_i) x_{pi}^k + \sum_{p=1}^{l} \sum_{h=1}^{r} \delta_{pm+h}(u_i, v_i)(x_{pi} - K_{ph})_+^m \tag{2}$$

and truncated function

$$\left(x_{pi} - K_{ph}\right)_+^m = \begin{cases} (x_{pi} - K_{ph}), x_{pi} \geq K_{ph} \\ 0, \quad x_{pi} < K_{ph} \end{cases}$$

Mathematically the relationship between the response variable y_i and the predictor variables $(x_{1i}, x_{2i}, \ldots, x_{li},)$ on the i-th location for nonparametric geographically weighted regression models with truncated spline approach, can be expressed as follows:

$$y_i = \beta_0(u_i, v_i) + \sum_{p=1}^{l} \sum_{k=1}^{m} \beta_{pk}(u_i, v_i) x_{pi}^k + \sum_{p=1}^{l} \sum_{h=1}^{r} \delta_{pm+h}(u_i, v_i)(x_{pi} - K_{ph})_+^m + \varepsilon_i \tag{3}$$

Equation (3) is a nonparametric geographically weighted regression models with truncated spline approach m-order with n number of areas. The components in Equation (3) are described as follows: y_i is respon variable to locations all i where $i = 1, 2, \ldots, n$. x_{pi} is predictor variables on the p- that i area with $p = 1, 2, \ldots, l$. K_{ph} is point knots on the h-th at the components of predictor variables p with $h = 1, 2, \ldots, r$. $\beta_{pk}(u_i, v_i)$ is parameter regression on the k- that the predictor variables p and the i-th area.

$\delta_{pm+h}(u_i, v_i)$ is parameter regression of the function truncated, this parameter is a parameter to $l + h$, at which point knots h-th and predictor variables p. Equation (3) can be expressed by:

$$\widetilde{\mathbf{Y}} = \tilde{f} + \tilde{\varepsilon} = \mathbf{X}\tilde{\beta}(u_i, v_i) + \mathbf{P}\,\tilde{\delta}(u_i, v_i) + \tilde{\varepsilon} \tag{4}$$

with

$$\widetilde{\boldsymbol{Y}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tilde{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\tilde{\beta}(u_i, v_i) = \begin{bmatrix} \beta_0(u_i, v_i) & \beta_{11}(u_i, v_i) & \beta_{12}(u_i, v_i) & \cdots & \beta_{l1}(u_i, v_i) & \cdots & \beta_{lm}(u_i, v_i) \end{bmatrix}^T$$

$$\tilde{\delta}(u_i, v_i) = \begin{bmatrix} \delta_{1m+1}(u_i, v_i) & \delta_{1m+2}(u_i, v_i) & \cdots & \delta_{lm+1}(u_i, v_i) & \cdots & \delta_{lm+r}(u_i, v_i) \end{bmatrix}^T$$

and the predictor variables defined by the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{11}^2 & x_{11}^3 & \cdots & x_{11}^m & x_{21} & x_{21}^2 & \cdots & x_{21}^m & \cdots & x_{l1} & x_{l1}^2 & \cdots & x_{l1}^m \\ 1 & x_{12} & x_{12}^2 & x_{12}^3 & \cdots & x_{12}^m & x_{22} & x_{22}^2 & \cdots & x_{22}^m & \cdots & x_{l2} & x_{l2}^2 & \cdots & x_{l2}^m \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & x_{1n}^2 & x_{1n}^3 & \cdots & x_{1n}^m & x_{2n} & x_{2n}^2 & \cdots & x_{22}^m & \cdots & x_{ln} & x_{ln}^2 & \cdots & x_{ln}^m \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} (x_{11} - K_{11})_+^m & \cdots & (x_{11} - K_{1r})_+^m & \cdots & (x_{l1} - K_{11})_+^m & \cdots & (x_{l1} - K_{1r})_+^m \\ (x_{12} - K_{11})_+^m & \cdots & (x_{12} - K_{1r})_+^m & \cdots & (x_{l2} - K_{11})_+^m & \cdots & (x_{l2} - K_{1r})_+^m \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (x_{1n} - K_{11})_+^m & \cdots & (x_{1n} - K_{1r})_+^m & \cdots & (x_{2n} - K_{1r})_+^m & \cdots & (x_{ln} - K_{1r})_+^m \end{bmatrix}$$

3. Results and Discussion

3.1 Estimation model

In nonparametric geographically weighted regression models with truncated spline approach, the assumptions used are error i-th observation of identical, independent and normally distributed with zero mean and variance $\sigma^2(u_i, v_i)$, where the parameter $\tilde{\beta}(u_i, v_i)$, $\tilde{\delta}(u_i, v_i)$ and $\sigma^2(u_i, v_i)$ to are unknown. This shows that every geographical location has different parameter values. If the parameter value is constant at any geographical location, the nonparametric geographically weighted regression models with truncated spline approach is the same as usual nonparametric regression model. That means that each geographical location has the same model.

Theorem 1.

If the regression model (3) with an error ε_i normally distributed with zero mean and variance $\sigma^2(u_i, v_i)$ was given Maximum Likelihood Estimator (MLE) is used to obtain estimator $\hat{\tilde{\beta}}(u_i, v_i), \hat{\tilde{\delta}}(u_i, v_i)$ and $\hat{\tilde{f}}$ as follows.

$$\hat{\tilde{\beta}}(u_i, v_i) = \mathbf{A(K)}\widetilde{\mathbf{Y}}$$
$$\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{B(K)}\widetilde{\mathbf{Y}}$$
$$\hat{\tilde{f}} = \mathbf{X}\hat{\tilde{\beta}}(u_i, v_i) + \mathbf{P}\,\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{C(K)}\widetilde{\mathbf{Y}}$$

with

$$\mathbf{A(K)} = \mathbf{S}(\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}[\mathbf{X}^T - \mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{P}(\mathbf{P}^T\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P}^T]\mathbf{W}(u_i, v_i)$$
$$\mathbf{B(K)} = \mathbf{R}(\mathbf{P}^T\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P}^T - \mathbf{P}^T\mathbf{W}(u_i, v_i)\mathbf{X}(\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X}^T]\mathbf{W}(u_i, v_i)$$
$$\mathbf{C(K)} = \mathbf{X}\,\mathbf{A(K)} + \mathbf{P}\mathbf{B(K)}$$

Proof:

Regression model given in Equation (3), Having obtained the joint density function of $y_1, y_2, \ldots, y_n$ then to estimate the nonparametric geographically weighted regression models with truncated spline approach on location ke- j, given the likelihood function on the location ke- j is as following:

$$L\left(\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j), \sigma^2(u_j, v_j)|Y\right) = (2\pi)^{-\frac{n}{2}}\left(\sigma^2(u_j, v_j)\right)^{-\frac{n}{2}}$$

$$\exp\left[-\frac{1}{2\sigma^2(u_j, v_j)}\sum_{i=1}^{n}\left[y_j - \left(\beta_0(u_j, v_j) + \sum_{p=1}^{l}\sum_{k=1}^{m}\beta_{pk}(u_j, v_j) x_{pi}^k + \sum_{p=1}^{l}\sum_{h=1}^{r}\delta_{pm+h}(u_j, v_j)(x_{pi} - K_{ph})_+^m\right)\right]^2\right] \tag{5}$$

In this research to gain estimator parameter $\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j), \sigma^2(u_j, v_j)$ of nonparametric geographically weighted regression models with truncated spline approach, it takes a geographic weighting. Data at each location of the observations are weighted depending on the size of influence between the observation locations. The closer the distance between observation locations, the greater is the influence of an observation location. Suppose the weighting for the location i and location j is $w_{i(j)}$, then to get the parameter estimator $\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j), \sigma^2(u_j, v_j)$ first providing j ke- weighting on the following observations:

$$w_{i(j)}y_i = w_{i(j)}\left[\beta_0(u_j, v_j) + \sum_{p=1}^{l}\sum_{k=1}^{m}\beta_{pk}(u_j, v_j) x_{pi}^k + \sum_{p=1}^{l}\sum_{h=1}^{r}\delta_{pm+h}(u_j, v_j)(x_{pi} - K_{ph})_+^m + \varepsilon_i\right]$$

$\varepsilon_i^* = w_{i(j)}(u_j, v_j)\varepsilon_i$, ε_i^* will follow a normal distribution with mean zero and variance $\sigma^2(u_j, v_j)$ and $\varepsilon_i^* \sim N\left(0, \sigma^2(u_j, v_j)\right)$. The weighted likelihood function is

$$L\left(\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j), \sigma^2(u_j, v_j)|Y\right) = (2\pi)^{-\frac{n}{2}}\left(\sigma^2(u_j, v_j)\right)^{-\frac{n}{2}}$$

$$\exp\left[-\frac{1}{2\sigma^2(u_j, v_j)}\sum_{i=1}^{n}w_{i(j)}(u_j, v_j)\left[y_j - \left(\beta_0(u_j, v_j) + \sum_{p=1}^{l}\sum_{k=1}^{m}\beta_{pk}(u_j, v_j) x_{pi}^k + \sum_{p=1}^{l}\sum_{h=1}^{r}\delta_{pm+h}(u_j, v_j)(x_{pi} - K_{ph})_+^m\right)\right]^2\right]$$

then performed the operation to facilitate the natural logarithm mathematical operations in order to obtain $\ln L$ equation as follows:

$$\ln L\left(\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j), \sigma^2(u_j, v_j)|Y\right) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\sigma^2(u_j, v_j)\right) - \frac{1}{2\sigma^2(u_j, v_j)}\boldsymbol{\Psi} \tag{6}$$

with

$$\boldsymbol{\Psi} = \sum_{i=1}^{n}w_{i(j)}\left[y_j - \left(\beta_0(u_j, v_j) + \sum_{p=1}^{l}\sum_{k=1}^{m}\beta_{pk}(u_j, v_j) x_{pi}^k + \sum_{p=1}^{l}\sum_{h=1}^{r}\delta_{pm+h}(u_j, v_j)(x_{pi} - K_{ph})_+^m\right)\right]^2$$

$$= \left(\tilde{\mathbf{Y}} - \mathbf{X}\,\tilde{\beta}(u_j, v_j) - \mathbf{P}\tilde{\delta}(u_j, v_j)\right)^T \mathbf{W}(u_j, v_j)\left(\tilde{\mathbf{Y}} - \mathbf{X}\,\tilde{\beta}(u_j, v_j) - \mathbf{P}\tilde{\delta}(u_j, v_j)\right)$$

Estimation parameter $\tilde{\beta}(u_j, v_j), \tilde{\delta}(u_j, v_j)$ and $\sigma^2(u_j, v_j)$ obtained by maximizing $\ln L$ shape Equation (6). Estimator $\tilde{\beta}(u_j, v_j)$ will be obtained based on the following derivatives:

$$\frac{\partial\boldsymbol{\Psi}}{\partial\tilde{\beta}(u_j, v_j)} = \frac{\partial\left[\left(\tilde{\mathbf{Y}} - \mathbf{X}\,\tilde{\beta}(u_j, v_j) - \mathbf{P}\tilde{\delta}(u_j, v_j)\right)^T \mathbf{W}(u_j, v_j)\left(\tilde{\mathbf{Y}} - \mathbf{X}\,\tilde{\beta}(u_j, v_j) - \mathbf{P}\tilde{\delta}(u_j, v_j)\right)\right]}{\partial\tilde{\beta}(u_j, v_j)}$$

$$= \frac{\partial\left[-2\left(\tilde{\beta}(u_j, v_j)\right)^T \mathbf{X}^T\mathbf{W}(u_j, v_j)\tilde{\mathbf{Y}} + 2\left(\tilde{\beta}(u_j, v_j)\right)^T \mathbf{X}^T\mathbf{W}(u_j, v_j)\mathbf{P}\tilde{\delta}(u_j, v_j) + \left(\tilde{\beta}(u_j, v_j)\right)^T \mathbf{X}^T\mathbf{W}(u_j, v_j)\mathbf{X}\,\tilde{\beta}(u_j, v_j)\right]}{\partial\tilde{\beta}(u_j, v_j)}$$

$$-2\,\mathbf{X}^T\mathbf{W}(u_j, v_j)\tilde{\mathbf{Y}} + 2\,\mathbf{X}^T\mathbf{W}(u_j, v_j)\mathbf{P}\tilde{\delta}(u_j, v_j) + 2\,\mathbf{X}^T\mathbf{W}(u_j, v_j)\mathbf{X}\,\tilde{\beta}(u_j, v_j) = 0$$

$$\mathbf{X^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \mathbf{X^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j) - \mathbf{X^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j) = 0$$

$$\mathbf{X^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j) = \mathbf{X^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \mathbf{X^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j)$$

$$\widetilde{\beta}(u_j,v_j) = \left(\mathbf{X^T W}(u_j,v_j)\mathbf{X}\right)^{-1}\mathbf{X^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \left(\mathbf{X^T W}(u_j,v_j)\mathbf{X}\right)^{-1}\mathbf{X^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j)$$

So that the parameter estimator $\widehat{\widetilde{\beta}}(u_j,v_j)$ is

$$\widehat{\widetilde{\beta}}(u_j,v_j) = \left(\mathbf{X^T W}(u_j,v_j)\mathbf{X}\right)^{-1}\mathbf{X^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \left(\mathbf{X^T W}(u_j,v_j)\mathbf{X}\right)^{-1}\mathbf{X^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j) \qquad (7)$$

because there are n observations location, then using Equation (7) obtained estimator $\widehat{\widetilde{\beta}}(u_i,v_i)$ is

$$\widehat{\widetilde{\beta}}(u_i,v_i) = (\mathbf{X^T\,W}(u_i,v_i)\,\mathbf{X})^{-1}\left[\mathbf{X^T\,W}(u_i,v_i)\,\widetilde{\mathbf{Y}} - \mathbf{X^T\,W}(u_i,v_i)\,\mathbf{P}\widehat{\widetilde{\delta}}(u_i,v_i)\right] \qquad (8)$$

Furthermore, to obtain estimator $\widehat{\widetilde{\delta}}(u_j,v_j)$, can be obtained by maximizing the form ln *L* Equation (6) and conducted operations against the derivative $\widetilde{\delta}(u_j,v_j)$.

$$\frac{\partial\mathbf{\Psi}}{\partial\widetilde{\delta}(u_j,v_j)} = \frac{\partial\left[\left(\widetilde{\mathbf{Y}} - \mathbf{X}\,\widetilde{\beta}(u_j,v_j) - \mathbf{P}\widetilde{\delta}(u_j,v_j)\right)^T \mathbf{W}(u_j,v_j)\left(\widetilde{\mathbf{Y}} - \mathbf{X}\,\widetilde{\beta}(u_j,v_j) - \mathbf{P}\widetilde{\delta}(u_j,v_j)\right)\right]}{\partial\widetilde{\delta}(u_j,v_j)}$$

$$= \frac{\partial\left[-2\left(\widetilde{\delta}(u_j,v_j)\right)^T \mathbf{P^T W}(u_j,v_j)\widetilde{\mathbf{Y}} + 2\left(\widetilde{\delta}(u_j,v_j)\right)^T \mathbf{P^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j) + \left(\widetilde{\delta}(u_j,v_j)\right)^T \mathbf{P^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j)\right]}{\partial\widetilde{\delta}(u_j,v_j)}$$

$$-2\,\mathbf{P^T W}(u_j,v_j)\widetilde{\mathbf{Y}} + 2\mathbf{P^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j) + 2\mathbf{P^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j) = 0$$

$$\mathbf{P^T W}(u_j,v_j)\mathbf{P}\widetilde{\delta}(u_j,v_j) = \mathbf{P^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \mathbf{P^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j)$$

$$\widetilde{\delta}(u_j,v_j) = \left(\mathbf{P^T W}(u_j,v_j)\mathbf{P}\right)^{-1}\mathbf{P^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \left(\mathbf{P^T W}(u_j,v_j)\mathbf{P}\right)^{-1}\mathbf{P^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j)$$

So that the parameter estimator $\widehat{\widetilde{\delta}}(u_j,v_j)$ is

$$\widehat{\widetilde{\delta}}(u_j,v_j) = \left(\mathbf{P^T W}(u_j,v_j)\mathbf{P}\right)^{-1}\mathbf{P^T W}(u_j,v_j)\widetilde{\mathbf{Y}} - \left(\mathbf{P^T W}(u_j,v_j)\mathbf{P}\right)^{-1}\mathbf{P^T W}(u_j,v_j)\mathbf{X}\,\widetilde{\beta}(u_j,v_j) \qquad (9)$$

because there are n observations location, then using Equation (9) obtained estimator $\widehat{\widetilde{\delta}}(u_i,v_i)$ is

$$\widehat{\widetilde{\delta}}(u_i,v_i) = (\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\left[\mathbf{P^T W}(u_i,v_i)\widetilde{\mathbf{Y}} - \mathbf{P^T W}(u_i,v_i)\mathbf{X}\,\widehat{\widetilde{\beta}}(u_i,v_i)\right] \qquad (10)$$

Estimator $\widehat{\widetilde{\beta}}(u_i,v_i)$ in Equation (8) still contains estimator $\widehat{\widetilde{\delta}}(u_i,v_i)$. Similarly estimator $\widehat{\widetilde{\delta}}(u_i,v_i)$ in Equation (10) still contains estimator $\widehat{\widetilde{\beta}}(u_i,v_i)$. In order to obtain the form of independent estimator it is necessary to substitution technique. To obtain the estimator $\widehat{\widetilde{\beta}}(u_i,v_i)$ free of $\widehat{\widetilde{\delta}}(u_i,v_i)$ the substitution of the Equation (8) to the Equation (10) as follows:

$$\widehat{\widetilde{\delta}}(u_i,v_i) = (\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\widetilde{\mathbf{Y}} - (\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\mathbf{X}\,\widehat{\widetilde{\beta}}(u_i,v_i).$$

$$= (\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\widetilde{\mathbf{Y}} +$$

$$-[(\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\mathbf{X}\{(\mathbf{X^T\,W}(u_i,v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i,v_i)\,\widetilde{\mathbf{Y}} +$$

$$-(\mathbf{X^T W}(u_i,v_i)\mathbf{X})^{-1}\mathbf{X^T\,W}(u_i,v_i)\,\mathbf{P}\widehat{\widetilde{\delta}}(u_i,v_i)\}]$$

$$= (\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\widetilde{\mathbf{Y}} +$$

$$-(\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\mathbf{X}((\mathbf{X^T\,W}(u_i,v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i,v_i)\,\widetilde{\mathbf{Y}}) +$$

$$+(\mathbf{P^T W}(u_i,v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i,v_i)\mathbf{X}\left((\mathbf{X^T W}(u_i,v_i)\mathbf{X})^{-1}\mathbf{X^T\,W}(u_i,v_i)\,\mathbf{P}\widetilde{\delta}(u_i,v_i)\right).$$

Part containing estimator $\hat{\tilde{\delta}}(u_i, v_i)$ grouped in a segment as follows:

$$\hat{\tilde{\delta}}(u_i, v_i) - (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{P}\tilde{\delta}(u_i, v_i) =$$
$$(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\widetilde{\mathbf{Y}} +$$
$$-(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}.$$

With a bit of elaboration of the following equation:

$$[\mathrm{I} - (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{P}]\hat{\tilde{\delta}}(u_i, v_i) =$$
$$(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\widetilde{\mathbf{Y}} +$$
$$-(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}.$$

Part that does not load estimator $\hat{\tilde{\delta}}(u_i, v_i)$ moved to the right

$$\hat{\tilde{\delta}}(u_i, v_i) = [\mathrm{I} - (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{P}]^{-1}$$
$$[(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\widetilde{\mathbf{Y}} +$$
$$-(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}]. \tag{11}$$

If defined

$$\mathbf{R} = [\mathrm{I} - (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{P}]^{-1} \tag{12}$$

By replacing the existing components in Equation (11) with R defined on similarities (12), Equation (11) can be written as follows:

$$\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{R}[(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\widetilde{\mathbf{Y}} +$$
$$-(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}].$$

Then the estimator obtained $\hat{\tilde{\delta}}(u_i, v_i)$

$$\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{R}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\widetilde{\mathbf{Y}} +$$
$$-\mathbf{R}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}$$
$$= \mathbf{R}\ (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P^T} - \mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}]\mathbf{W}(u_i, v_i)\widetilde{\mathbf{Y}}$$
$$= \mathbf{B(K)}\widetilde{\mathbf{Y}}$$

Thus obtained estimator $\hat{\tilde{\delta}}(u_i, v_i)$ is

$$\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{B(K)}\widetilde{\mathbf{Y}} \tag{13}$$

With hat matrix $\mathbf{B(K)}$ is

$$\mathbf{B(K)} = \mathbf{R}\ (\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P^T} - \mathbf{P^T W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}\ \mathbf{X^T}]\mathbf{W}(u_i, v_i)$$

Next to gain estimator $\hat{\tilde{\beta}}(u_i, v_i)$ free of $\hat{\tilde{\delta}}(u_i, v_i)$ the substitution of the Equation (10) to the Equation (8). The obtained estimator $\hat{\tilde{\beta}}(u_i, v_i)$ is

$$\hat{\tilde{\beta}}(u_i, v_i) = \mathbf{S}(\mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{X})^{-1}[\ \mathbf{X^T} - \mathbf{X^T}\ \mathbf{W}(u_i, v_i)\ \mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}]\mathbf{W}(u_i, v_i)\ \widetilde{\mathbf{Y}}$$
$$= \mathbf{A(K)}\widetilde{\mathbf{Y}}$$

Based on the description above estimator $\hat{\tilde{\beta}}(u_i, v_i)$ can be written as follows:

$$\hat{\tilde{\beta}}(u_i, v_i) = \mathbf{A(K)}\tilde{\mathbf{Y}} \tag{14}$$

With the hat matrix $\mathbf{A(K)}$ is

$$\mathbf{A(K)} = \mathbf{S}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}[\,\mathbf{X^T} - \mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}]\mathbf{W}(u_i, v_i)$$

Next to determine the estimator function of nonparametric geographically weighted regression with truncated spline approach in Equation (2), can be substituted $\hat{\tilde{\beta}}(u_i, v_i)$ and $\hat{\tilde{\delta}}(u_i, v_i)$ value in the following equation:

$$
\begin{aligned}
\hat{\tilde{f}} &= \mathbf{X}\hat{\tilde{\beta}}(u_i, v_i) + \mathbf{P}\,\hat{\tilde{\delta}}(u_i, v_i)\\
&= \mathbf{X}\,\mathbf{A(K)}\tilde{\mathbf{Y}} + \mathbf{PB(K)}\tilde{\mathbf{Y}}\\
&= [\mathbf{X}\,\mathbf{A(K)} + \mathbf{PB(K)}]\tilde{\mathbf{Y}}\\
&= \mathbf{C(K)}\tilde{\mathbf{Y}}
\end{aligned}
$$

The matrix $\mathbf{C(K)} = \mathbf{X\,A(K)} + \mathbf{PB(K)}$ is a hat matrix containing knots point K for nonparametric geographically weighted regression models with truncated spline approach.

3.2 Unbiased Estimator $\hat{\tilde{\beta}}(u_i, v_i)$, $\hat{\tilde{\delta}}(u_i, v_i)$ and $\hat{\tilde{f}}$

Unbiased estimator $\hat{\tilde{\beta}}(u_i, v_i)$ is given in lemma 1 as follows:

Lemma 1.

If $\hat{\tilde{\beta}}(u_i, v_i)$ is a estimator of nonparametric geographically weighted regression with truncated spline approach that follows the Equation (3), then $\hat{\tilde{\beta}}(u_i, v_i)$ is an unbiased estimator for $\tilde{\beta}(u_i, v_i)$.

Proof:

After the obtained estimator $\tilde{\beta}(u_i, v_i)$, then shown an unbiased nature of the estimator $\tilde{\beta}(u_i, v_i)$ in the following manner:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\beta}}(u_i, v_i)\right) &= \mathrm{E}\left(\mathbf{A(K)}\tilde{\mathbf{Y}}\right).\\
&= \mathrm{E}\left(\mathbf{S}(\mathbf{X^T}\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}[\,\mathbf{X^T} - \mathbf{X^T}\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}]\mathbf{W}(u_i, v_i)\tilde{\mathbf{Y}}\right).\\
&= \left(\mathbf{S}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i) + \right.\\
&\quad \left. -\mathbf{S}(\mathbf{X^T}\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}\mathbf{W}(u_i, v_i)\right)\mathrm{E}\left(\tilde{\mathbf{Y}}\right). \tag{15}
\end{aligned}
$$

Furthermore $\mathrm{E}\left(\tilde{\mathbf{Y}}\right)$ is substituted into the Equation (25), in order to obtain the following equation:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\beta}}(u_i, v_i)\right) &= \left(\mathbf{S}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i) + \right.\\
&\quad \left. -\mathbf{S}(\mathbf{X^T}\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}\mathbf{W}(u_i, v_i)\right)\\
&\quad \left(\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \mathbf{P}\,\tilde{\delta}(u_i, v_i)\right).
\end{aligned}
$$

With a bit of mathematical elaboration and there are some components of the matrix is equal to 1, was obtained

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\beta}}(u_i, v_i)\right) &= \mathbf{S}\,\tilde{\beta}(u_i, v_i) + \mathbf{S}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{P}\,\tilde{\delta}(u_i, v_i) + \\
&\quad -\mathbf{S}(\mathbf{X^T}\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}\mathbf{W}(u_i, v_i)\,\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \\
&\quad -\mathbf{S}(\mathbf{X^T}\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}\,\tilde{\delta}(u_i, v_i).
\end{aligned}
$$

Furthermore, the elaboration of mathematical is obtained:

$$\mathrm{E}\left(\hat{\tilde{\beta}}(u_i, v_i)\right) = \mathbf{S}\,\tilde{\beta}(u_i, v_i) + $$

$$-S(X^T W(u_i, v_i)X)^{-1}X^T \ W(u_i, v_i) \ P(P^T W(u_i, v_i)P)^{-1}P^T W(u_i, v_i) \ X \ \tilde{\beta}(u_i, v_i).$$

$$= [I - (X^T W(u_i, v_i)X)^{-1}X^T \ W(u_i, v_i) \ P(P^T W(u_i, v_i)P)^{-1}P^T W(u_i, v_i) \ X]S\tilde{\beta}(u_i, v_i).$$

Furthermore substitutable matrix S, is obtained:

$$E\left(\hat{\tilde{\beta}}(u_i, v_i)\right) = [I - (X^T W(u_i, v_i)X)^{-1}X^T \ W(u_i, v_i) \ P(P^T W(u_i, v_i)P)^{-1}P^T W(u_i, v_i) \ X]$$
$$[I - (X^T W(u_i, v_i)X)^{-1}X^T \ W(u_i, v_i) \ P(P^T W(u_i, v_i)P)^{-1}P^T W(u_i, v_i) \ X]^{-1}\tilde{\beta}(u_i, v_i).$$

Thus obtained

$$E\left(\hat{\tilde{\beta}}(u_i, v_i)\right) = \tilde{\beta}(u_i, v_i)$$

From the results obtained have proved $\hat{\tilde{\beta}}(u_i, v_i)$ is an unbiased estimator $\tilde{\beta}(u_i, v_i)$ ∎

The unbiased estimator $\hat{\tilde{\delta}}(u_i, v_i)$ and estimator $\hat{\tilde{f}}$ is given in lemma 2 and lemma 3, Proof of lemma 2 and lemma 3 in Appendix.

Lemma 2.

If $\hat{\tilde{\delta}}(u_i, v_i)$ is estimator of nonparametric geographically weighted regression models with truncated spline approach that follows the Equation (3), then $\hat{\tilde{\delta}}(u_i, v_i)$ is an unbiased estimator for $\tilde{\delta}(u_i, v_i)$.

Lemma 3.

If $\hat{\tilde{f}}$ is the estimator function of nonparametric geographically weighted regression models with truncated spline approach that follows the Equation (3), then $\hat{\tilde{f}}$ is an unbiased estimator for $\tilde{f}$.

4. Discussion

Analysis of the data on the crude birth rate with five predictor variables i.e. the percentage of family head education (x_1: completed / not completed for the primary school), the percentage of working status of family head (x_2: having work / no work), the percentage of family head who married at 15-19 years old (x_3), the number of rough marriages (x_4) and the number of migrant (x_5). Selection of the optimum knot points of the nonparametric regression methods and nonparametric geographically weighted regression methods with truncated spline approach is respectively given in Table 1 and Table 2.

Estimation using the Nonparametric Regression Geographically Weighted Regression with Truncated Spline Approach is as follows:

$$\hat{y} = 29.5 - 0.59x_1 + 5.71(x_1 - 1.93)_+ - 134(x_1 - 3.13)_+ + 133(x_1 - 3.23)_+$$
$$-7.57x_2 + 9.59(x_2 - 2.15)_+ - 16.9(x_2 - 3.56)_+ + 14.4(x_2 - 3.66)_+ +$$
$$-48.9x_3 + 140(x_3 - 0.12)_+ - 940(x_3 - 0.20)_+ + 856(x_3 - 0.21)_+ +$$
$$-1.33x_4 + 1.68(x_4 - 2.46)_+ - 11.3(x_4 - 5.13)_+ - 12.3(x_4 - 5.34)_+ +$$
$$+0.136x_5 - 0.273(x_5 - 31.63)_+ + 6.59(x_5 - 47)_+ - 6.78(x_5 - 48.18)_+$$

Nonparametric Geographically Weighted Regression with Truncated Spline Approach with 3 knot points that has 97.3% of R^2. This may explain the crude birth rate of 97.3%.

Table 1.   Optimum knot point of nonparametric regression method with truncated spline approach.

Number of knots points	x_1	x_2	x_3	x_4	x_5	GCV
3	1.93	1.18	6.47	2.46	31.63	1.86
	3.13	0.07	0.35	5.13	47	
	3.23	3.67	0.23	5.34	48.18	

Table 2.   Optimum knot point of nonparametric regression geographically weighted regression with truncated spline approach.

Number of knots points	x_1	x_2	x_3	x_4	x_5	GCV
3	1.93	2.15	0.12	2.46	31.63	1.38
	3.13	3.56	0.20	5.13	47.00	
	3.23	3.66	0.21	5.34	48.18	

5. Conclusions

Estimation of nonparametric geographically weighted regression using truncated spline approach was successfully formulated. It was found that:

1. Nonparametric geographically weighted regression models using truncated spline approach is $\widetilde{Y} = \tilde{f} + \tilde{\varepsilon}$ with $\tilde{f} = \mathbf{X}\tilde{\beta}(u_i, v_i) + \mathbf{P}\,\tilde{\delta}(u_i, v_i)$, obtained estimator $\hat{\tilde{\beta}}(u_i, v_i) = \mathbf{A}(\mathbf{K})\widetilde{Y}$ with matrix hat $\mathbf{A}(\mathbf{K})$ is

$$\mathbf{A}(\mathbf{K}) = \mathbf{S}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}[\,\mathbf{X^T} - \mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}]\mathbf{W}(u_i, v_i).$$

and estimator $\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{B}(\mathbf{K})\widetilde{Y}$ with matrix hat $\mathbf{B}(\mathbf{K})$ is

$$\mathbf{B}(\mathbf{K}) = \mathbf{R}\,(\mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P^T} - \mathbf{P^T}\mathbf{W}(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}]\mathbf{W}(u_i, v_i).$$

Regression curve Estimator is $\hat{\tilde{f}} = \mathbf{X}\hat{\tilde{\beta}}(u_i, v_i) + \mathbf{P}\,\hat{\tilde{\delta}}(u_i, v_i) = \mathbf{C}(\mathbf{K})\widetilde{Y}$. with matrix hat $\mathbf{C}(\mathbf{K}) = \mathbf{X}\,\mathbf{A}(\mathbf{K}) + \mathbf{PB}(\mathbf{K})$ is a matrix that has a point knots $\mathbf{K}$.

2. Result of estimators $\hat{\tilde{\beta}}(u_i, v_i)$, $\hat{\tilde{\delta}}(u_i, v_i)$ and $\hat{\tilde{f}}$ from nonparametric geographically weighted regression models using truncated spline approach has been shown as unbiased estimator.

3. Application of nonparametric geographically weighted regression models with truncated spline approach of the crude birth rate data in 38 areas in eastern Java, have resulted GCV values smaller than when using a nonparametric regression method. It concluded that using the nonparametric geographically weighted regression models with truncated spline approach to model the effect of the crude birth rate was better and more appropriately than the nonparametric regression models with truncated spline approach.

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Appendix

Proof of Lemma 2.

After the obtained estimator $\tilde{\delta}(u_i, v_i)$, then shown an unbiased nature of the estimator $\tilde{\delta}(u_i, v_i)$ in the following manner:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= \mathrm{E}(\mathbf{B(K)}\tilde{\mathbf{Y}}). \\
&= \mathrm{E}\big[\mathbf{R}\,(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P^T} - \mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}]W(u_i, v_i)\tilde{\mathbf{Y}}\big]. \\
&= \mathrm{E}\big(\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\tilde{\mathbf{Y}} + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\,\tilde{\mathbf{Y}}\big). \\
&= \big(\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\big)\mathrm{E}(\tilde{\mathbf{Y}}).
\end{aligned}
\tag{16}
$$

Furthermore $\mathrm{E}(\tilde{\mathbf{Y}})$ is substituted into the Equation (16), in order to obtain the following equation:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= \big(\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\big) \\
&\quad \big(\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \mathbf{P}\,\tilde{\delta}(u_i, v_i)\big).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= \mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \\
&\quad +\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{P}\,\tilde{\delta}(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}\,\tilde{\delta}(u_i, v_i).
\end{aligned}
\tag{17}
$$

There are several components of the matrix is equal to 1 so that the Equation (17) can be simplified as follows:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= \mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \mathbf{R}\tilde{\delta}(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}\,\tilde{\delta}(u_i, v_i).
\end{aligned}
$$

With a bit of mathematical elaboration:

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= \mathbf{R}\tilde{\delta}(u_i, v_i) + \\
&\quad -\mathbf{R}(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}\,\tilde{\delta}(u_i, v_i).
\end{aligned}
$$

Tribe containing $\tilde{\delta}(u_i, v_i)$ are grouped on the right, so that the Equation (27) can be simplified as follows:

$$
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) = [\mathrm{I}-(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}]\mathbf{R}\,\tilde{\delta}(u_i, v_i).
$$

Substitutable matrix **R** obtained

$$
\begin{aligned}
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) &= [\mathrm{I}-(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}] \times \\
&\quad [\mathrm{I}-(\mathbf{P^T}W(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}W(u_i, v_i)\mathbf{X}(\mathbf{X^T}\,W(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}\,W(u_i, v_i)\mathbf{P}]^{-1}\tilde{\delta}(u_i, v_i).
\end{aligned}
$$

Thus obtained

$$
\mathrm{E}\left(\hat{\tilde{\delta}}(u_i, v_i)\right) = \tilde{\delta}(u_i, v_i)
$$

From the above results proved that $\hat{\tilde{\delta}}(u_i, v_i)$ is an unbiased estimator $\tilde{\delta}(u_i, v_i)$ ∎

Proof of Lemma 3.

Having obtained estimator $\hat{\tilde{f}}$ models will further shown no bias nature of the estimator $\tilde{f}$ in the following way:

$$\begin{aligned}
\mathrm{E}\left(\hat{\tilde{f}}\right) &= \mathrm{E}\left(\mathbf{X}\hat{\tilde{\beta}}(u_i, v_i) + \mathbf{P}\hat{\tilde{\delta}}(u_i, v_i)\right) \\
&= \mathrm{E}\left(\mathbf{X}\,\mathbf{A(K)}\widetilde{\mathbf{Y}} + \mathbf{PB(K)}\widetilde{\mathbf{Y}}\right) \\
&= \mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) + \mathbf{PB(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right)
\end{aligned}$$

To prove the nature of unbias in the estimator $\hat{\tilde{f}}$, first completed component $\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right)$.

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) &= \mathbf{X}\,\mathbf{S}(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}[\,\mathbf{X^T} - \mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T}]\mathbf{W}(u_i, v_i)\mathrm{E}\left(\widetilde{\mathbf{Y}}\right). \\
&= \Big(\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i) + \\
&\quad -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\Big)\mathrm{E}\left(\widetilde{\mathbf{Y}}\right).
\end{aligned}$$

Furthermore substitutable value of E (Y) is obtained:

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) &= \Big(\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i) + \\
&\quad -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\Big) \\
&\quad \Big(\mathbf{X}\,\tilde{\beta}(u_i, v_i) + \mathbf{P}\,\tilde{\delta}(u_i, v_i)\Big).
\end{aligned}$$

Based on the operating results of mathematics is obtained:

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) =\ & \mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) + \\
& +\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i).
\end{aligned}$$

There matrix components $(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{X} = 1$ and $(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{P} = 1$ thus obtained

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) =\ & \mathbf{X}\,\mathbf{S}\,\tilde{\beta}(u_i, v_i) + \mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}\tilde{\delta}(u_i, v_i).
\end{aligned}$$

With matrix mathematics operations obtained:

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) =\ & \mathbf{X}\,\mathbf{S}\,\tilde{\beta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) + \\
& +\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{X}\,)^{-1}\,\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}\,(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}\tilde{\delta}(u_i, v_i).
\end{aligned}$$

The equation is simplified as follows:

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) =\ & \mathbf{X}\,\mathbf{S}\,\tilde{\beta}(u_i, v_i) + \\
& -\mathbf{X}\,\mathbf{S}(\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i).
\end{aligned}$$

Thus obtained $\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right)$:

$$\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) = [\mathrm{I} - (\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}]\mathbf{X}\,\mathbf{S}\,\tilde{\beta}(u_i, v_i).$$

Substitutable matrix S obtained

$$\begin{aligned}
\mathbf{X}\,\mathbf{A(K)}\mathrm{E}\left(\widetilde{\mathbf{Y}}\right) =\ & [\mathrm{I} - (\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}]\mathbf{X} \\
& [\mathrm{I} - (\mathbf{X^T W}(u_i, v_i)\mathbf{X}\,)^{-1}\mathbf{X^T}\,\mathbf{W}(u_i, v_i)\,\mathbf{P}(\mathbf{P^T W}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^T W}(u_i, v_i)\mathbf{X}]^{-1}\tilde{\beta}(u_i, v_i).
\end{aligned}$$

Thus obtained

$$\mathbf{X\,A(K)}E(\widetilde{\mathbf{Y}}) = \mathbf{X}\,\tilde{\beta}(u_i, v_i) \tag{18}$$

Having obtained the value of the component $\mathbf{X\,A(K)}E(\widetilde{\mathbf{Y}})$, then resolved component $\mathbf{PB(K)}E(\widetilde{\mathbf{Y}})$ as follows:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) =$
$\quad = \mathbf{PR(P^TW}(u_i, v_i)\mathbf{P})^{-1}[\mathbf{P^T} - \mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T}]\mathbf{W}(u_i, v_i)E(\widetilde{\mathbf{Y}}).$
$\quad = \mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i) +$
$\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)E(\widetilde{\mathbf{Y}}).$

Furthermore substitutable value of $E(\widetilde{\mathbf{Y}})$ is obtained:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = \quad \mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i) +$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)$
$\quad\quad\quad\quad\quad\;\; \left(\mathbf{X}\tilde{\beta}(u_i, v_i) + \mathbf{P}\tilde{\delta}(u_i, v_i)\right).$

Based on the operating results of mathematics is obtained:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = \;\; \mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) +$
$\quad\quad\quad\quad\quad\;\; + \mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i) +$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i)$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i).$

There matrix components $(\mathbf{X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)\mathbf{X} = 1$ and $(\mathbf{P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{P} = 1$ thus obtained:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = \;\; \mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i) + \;\mathbf{P\,R}\tilde{\delta}(u_i, v_i) +$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X}\tilde{\beta}(u_i, v_i)$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i).$

The equation is simplified as follows:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = \mathbf{P\,R}\tilde{\delta}(u_i, v_i) +$
$\quad\quad\quad\quad\quad\;\; -\mathbf{P\,R(P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)\mathbf{P}\tilde{\delta}(u_i, v_i).$

Thus obtained $\mathbf{X\,A(K)}E(\widetilde{\mathbf{Y}})$:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = [\mathrm{I}-(\mathbf{P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)]\mathbf{P\,R}\tilde{\delta}(u_i, v_i).$

Substitutable matrix $\mathbf{R}$ obtained:

$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = [\mathrm{I}-(\mathbf{P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)]\mathbf{P}$
$\quad\quad\quad\quad\quad\;\; [\mathrm{I}-(\mathbf{P^TW}(u_i, v_i)\mathbf{P})^{-1}\mathbf{P^TW}(u_i, v_i)\mathbf{X(X^T\,W}(u_i, v_i)\,\mathbf{X})^{-1}\,\mathbf{X^T\,W}(u_i, v_i)]^{-1}\tilde{\delta}(u_i, v_i).$

Thus obtained

$$\mathbf{PB(K)}E(\widetilde{\mathbf{Y}}) = \mathbf{P}\tilde{\delta}(u_i, v_i) \tag{19}$$

Based on the Equation (18) and Equation (19) then the result of the expectation $\hat{\tilde{f}}$ is

$\mathrm{E}\left(\hat{\tilde{f}}\right) = \mathbf{X\,A(K)}E(\widetilde{\mathbf{Y}}) + \mathbf{PB(K)}E(\widetilde{\mathbf{Y}})$
$\quad\quad\; = \mathbf{X}\,\tilde{\beta}(u_i, v_i) + \mathbf{P}\tilde{\delta}(u_i, v_i)$
$\quad\quad\; = \tilde{f}$

So it proved not biased nature of the model estimator $\hat{\tilde{f}}$ ∎